COMPLEX NUMBER

EXERCISE – 1(A)

Q.1 [B]

$\Rightarrow \sqrt{-2}\sqrt{-3} = \left(i\sqrt{2}\right)\left(i\sqrt{3}\right) = -\sqrt{6}$

Q.2 [D]

$\Rightarrow (1+i)^5(1-i)^5 = \left(1-i^2\right)^5 = 2^5$

Q.3 [B]

$(1+i)^4 + (1-i)^4 = \left[(1+i)^2 + (1-i)^2\right]^2 - 2(1+i)^2(1-i)^2$

$\Rightarrow \left[(2i)+(-2i)\right]^2 - 2\left(1-i^2\right)^2$

$\Rightarrow -2.2^2 = -8$

Q.4 [C]

$(1+i)^8 + (1-i)^8 = \left[(1+i)^4 + (1-i)^4\right]^2 - 2(1+i)^4(1-i)^4$

$\Rightarrow [-8]^2 - 2\left(1-i^2\right)^4$

$\Rightarrow 64 - 2(2)^4 = 32$

Q.5 [A]

$(1+i)^6 + (1-i)^6 = \left[(1+i)^3 + (1-i)^3\right]^2 - 2(1+i)^3(1-i)^3$

$\Rightarrow \left[1+3i^2+3i+i^3+1-i^3+3i^2-3i\right]^2 - 2\left(1-i^2\right)^3$

$\Rightarrow [2-6]^2 - 2(2)^3$

Q.6 [A]

$\Rightarrow (1+i)^{10} = \left[(1+i)^2\right]^5 = (2i)^5 = 32i$

Q.7 [A]

$\Rightarrow 1+i^2+i^3-i^6+i^8 = 1-1-i-i^2+1 = 2-i$

Q.8 [B]

$\because i^4 = 1$

$\Rightarrow \therefore i^{4n+\lambda} = i^{\lambda}$ where $n \in I$; $i^{4n+2} = i^2 = -1$

$\therefore$ given expression will become

$$\Rightarrow \frac{1-1+1-1+1}{-1+1-1+1-1} - 1 = -2$$

Q.9 [D]

For given equation to be true

$(1-i)^n = 2^n$

$\Rightarrow n = 4m;\ m \in I$

$\Rightarrow \min n = 4$

Q.10 [A]

$$\left(\frac{-1+i}{1+i}\right)^n = \text{Real number}$$

$$\Rightarrow \left(\frac{-1+i}{1+i}\right)^n \left(\frac{1-i}{1-i}\right)^n = \frac{(1-i)^{2n}(-1)^n}{(1+1)^n} = \frac{\left(1+i^2-2i\right)^n(-1)^n}{2^n}$$

$$\Rightarrow \frac{2^n i^n}{2^n} = i^n$$

least n = 2

Q.11 [B]

$(a+ib)^5 = \alpha + i\beta$

$\Rightarrow i^5(-ai+b)^5 = \alpha + i\beta$

$\Rightarrow (b-ai)^5 = \beta - \alpha i$

Take complex conjugate then

$\Rightarrow (b+ai)^5 = \beta + \alpha i$

Q.12 [B]

$\frac{1+2i}{1-i}$

$\Rightarrow \frac{1+2i}{1-i} \times \frac{1+i}{1+i}$

$\Rightarrow \frac{1+3i+2i^2}{2}$

$\Rightarrow \frac{-1+3i}{2}$ 2^{nd} quadrate

Q.13 [A]

$|z| = 1,\ w = \frac{z-1}{z+1}$ $(z \neq -1)$

Let $z = \cos\theta + i\sin\theta$

$\Rightarrow \therefore w = \frac{(\cos\theta - 1) + i\sin\theta}{(\cos\theta + 1) + i\sin\theta} \neq 1$

$\Rightarrow \frac{-2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}$

$\Rightarrow \frac{2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}\left(\frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}\right)$

$\Rightarrow w = i\tan\frac{\theta}{2}$

$\Rightarrow \therefore \text{Re}(w) = 0$

Q.14 [C]

$$\frac{3+2i\sin\theta}{1-2i\sin\theta} = Ki; K \in R$$

$$\Rightarrow \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta} = Ki$$

$$\Rightarrow \therefore \frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0 \ \ (\text{Real part zero})$$

$$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 = \sin^2 60^o$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}$$

Q.15 [B]

Given $x = \frac{1}{x}$

$$\Rightarrow x = \pm 1$$

Q.16 [C]

z = 1 + i

$$\Rightarrow z^2 = 1 + i^2 + 2i = 2i$$

Let z_1 is multiplication inverse

$$\Rightarrow \therefore z^2 z_1 = 1$$

$$\Rightarrow z_1 = \frac{1}{z^2} = \frac{1}{zi} = \frac{-i}{z}$$

Q.17 [B]

$$(x+iy)^{\frac{1}{3}} = a + ib$$

$$\Rightarrow x + iy = (a+ib)^3$$

$$\Rightarrow x = a^3 - 3ab^2$$

$$\Rightarrow y = -b^3 + 3a^2b$$

$$\Rightarrow \therefore \frac{x}{a} + \frac{y}{b} = \left(a^2 - 3b^2\right) + \left(-b^2 + 3a^2\right)$$

$$\Rightarrow 4\left(a^2 - b^2\right)$$

Q.18 [B]

$$\sqrt{3} + i = (a + ib)(c + id)$$

$$\Rightarrow \arg\left(\sqrt{3} + i\right) = \arg\left[(a + ib)(c + id)\right]$$

$$\Rightarrow \arg(a + ib) + \arg(c + id) = \tan^{-1}\frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan^{-1}\frac{b}{a} + \tan^{-1}\frac{d}{c} = \frac{\pi}{6}$$

Q.19 [C]

$$z_1 = 4 + i5$$

$$\Rightarrow z_2 = -3 + 2i$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{4 + 5i}{-3 + 2i} = \frac{(4 + 5i)(-3 - 2i)}{9 + 4} = \frac{-12 + 10 - 15i - 8i}{13}$$

$$\Rightarrow \frac{-2}{13} - \frac{23}{13}i$$

Q.20 [C]

$$x + \frac{1}{x} = 2\cos\theta$$

$$\Rightarrow x^2 - 2x\cos\theta + 1 = 0$$

$$\Rightarrow x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$\Rightarrow x = \cos\theta \pm i\sin\theta$

Q.21 [A]

$3 - 2yi = 9^x - 7i$

$\Rightarrow 3 = 9^x ; \qquad -2y = -7$

$\Rightarrow x = \frac{1}{2}$

$\Rightarrow y = \frac{7}{2}$

Q.22 [B]

$\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$

$\Rightarrow \left((1+i)x - 2i(3-i) + \left[(2-3i)y + i\right][3+i]\right) = i(3+1)$

by comparing real & imaginary parts

$\Rightarrow$ x = 3, y = – 1

Q.23 [B]

$x + iy = \frac{3}{2+\cos\theta + i\sin\theta} \times \left(\frac{2+\cos\theta - i\sin\theta}{2+\cos\theta - i\sin\theta}\right)$

$\Rightarrow \frac{6 + 2\cos\theta - 3i\sin\theta}{\left(4 + \cos^2\theta + 4\cos\theta + \sin^2\theta\right)}$

$\Rightarrow x = \frac{2(3+\cos\theta)}{5+4\cos\theta},\ y = \frac{-3\sin\theta}{5+4\cos\theta}$

$\Rightarrow x^2 + y^2 = \frac{4\left(9 + \cos^2\theta + 6\cos\theta\right) + 9\sin^2\theta}{(5+4\cos\theta)^2} = \frac{40 + 2y\cos\theta + 5\sin^2\theta}{(5+4\cos\theta)^2}$

$\Rightarrow \frac{8(3+\cos\theta)(5+4\cos\theta) - 3(5+4\cos\theta)^2}{(5+4\cos\theta)^2}$

$\Rightarrow x^2 + y^2 = 4x - 3$

Q.24 [B]

$x = -5 + 2\sqrt{-4} = -5 + 4i$

$\Rightarrow x^2 - (-10)x + (41) = 0$

$\Rightarrow x^2 + 10x + 41 = 0$

$\Rightarrow x^2 = -10x - 41$

$\Rightarrow x^3 = -10x^2 - 41x = -10(-10x - 41) - 41x = 59x + 410$

$\Rightarrow \therefore x^4 + 9x^3 + 35x^2 - x + 4 = x^2(x^2 + 35) + 9x^3 - x + 4$

$\Rightarrow x^2(-10x - 6) + 9x^3 - x + 4$

$\Rightarrow -x^3 - 6x^2 - x + 4$

$\Rightarrow -59x + 410 + 60x + 41 \times 6 - x + 4$

$\Rightarrow -41 \times 4 + 4$

$\Rightarrow -160$

Q.25 [B]

(x + iy) (y – i3) = 4 + i

By comparing real & imaginary parts.

$\Rightarrow 2x + 3y = 4$ ………..(1)

$\Rightarrow 2y - 3x = 1$ ………..(2)

$\Rightarrow \therefore x = \frac{5}{13}, y = \frac{14}{13}$

Q.26 [D]

$z = \frac{1 - i\sin\alpha}{1 + 2i\sin\alpha} = k;\ \ k \in R$

$$\Rightarrow z=\frac{(1-i\sin\alpha)(1-2i\sin\alpha)}{1+4\sin^2\alpha}$$

$$\Rightarrow \therefore \operatorname{Im}(z)=0$$

$$\Rightarrow -2\sin\alpha-\sin\alpha=0$$

$$\Rightarrow \alpha=n\pi;\quad n\in I$$

Q.27 [C]

$$z(2-i)=3+i$$

$$\Rightarrow z=\frac{3+i}{2-i}\times\frac{2+i}{2+i}=\frac{6-1+5i}{5}$$

$$\Rightarrow z=1+i=\sqrt{2}e^{\frac{i\pi}{4}}$$

$$\Rightarrow z^{20}=2^{10}e^{i5\pi}=2^{10}=1024$$

Q.28 [A]

$$\operatorname{Re}\left(\frac{z-8i}{z+6}\right)=0$$

$$\Rightarrow z=\frac{z+i(y-8)}{(x+6)+iy}=\frac{[x+i(y-8)][(x+6)-iy]}{(x+6)^2+y^2}$$

$$\Rightarrow \operatorname{Re}(z)=x(x+6)+y(y-8)=0$$

$$\Rightarrow x^2+y^2+6x-8y=0$$

Q.29 [B]

$$z=\frac{2+5i}{4-3i}\times\frac{4+3i}{4+3i}=\frac{8-15+26i}{25}$$

$$\Rightarrow z=\frac{-7}{25}+\frac{i26}{25}$$

$$\Rightarrow \bar{z}=\frac{-7}{25}-\frac{i26}{25}$$

Q.30 [B]

$z_1 + z_2 = \text{Real}$

$\Rightarrow z_1 z_2 = \text{Real}$

$\Rightarrow \therefore z_1 \,\&\, z_2$ are complex conjugate

$\Rightarrow z_1 = \overline{z_2}$

Q.31 [B]

z = x + iy (in 3rd quadrate)

$\Rightarrow x < 0,\ y < 0$

$\Rightarrow \bar{z} = x - iy = x + i(y)$ 2nd quadrate

Q.32 [A]

$(z+3)(\bar{z}+3)$

$\Rightarrow (z+3)(\overline{z+3})$

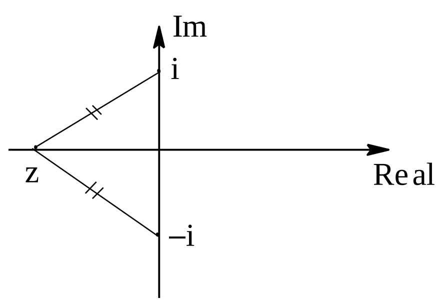
$\Rightarrow |z+3|^2$

Q.33 [A]

$\Rightarrow |z_1 z_2| = |z_1||z_2| = 1$

Q.34 [A]

$|z+1| = |z-i|$



Locus of z wier be Real axis

Q.35 [B]

$$\frac{z-1}{z+1}=ki;k\in R$$

$$\Rightarrow \text{Re}\left[\frac{(x-1)+iy}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}\right]=0$$

$$\Rightarrow (x-1)(x+1)+y^2=0$$

$$\Rightarrow x^2+y^2=1$$

$$\Rightarrow |z|^2=1$$

$$\Rightarrow |z|=1$$

Q.36

$$\Rightarrow |2z-1|+|3z-2|$$

Q.37 [B]

$$|z_1+z_2|^2+|z_1-z_2|^2$$

$$\Rightarrow (z_1+z_2)(\overline{z_1}+\overline{z_2})+(z_1-z_2)(\overline{z_1}-\overline{z_2})$$

$$\Rightarrow 2\left(|z_1|^2+|z_2|^2\right)$$

Q.38

$$\frac{2z_1}{3z_2}=ki;k\in R$$

$$\Rightarrow \frac{2z_1\overline{z_2}}{3|z_2|^2}=ki$$

$$\Rightarrow \therefore \text{Re}\left(z_1\overline{z_2}\right)=0$$

$$\Rightarrow z_1\overline{z_2}+\overline{z_1}z_2=0$$

$$\Rightarrow \therefore \left|\frac{z_1 - z_2}{z_1 + z_2}\right|^2 = \left(\frac{z_1 - z_2}{z_1 + z_2}\right)\left(\frac{\overline{z_1} - \overline{z_2}}{\overline{\overline{z_1} + \overline{z_2}}}\right)$$

$$\Rightarrow \frac{|z_1|^2 + |z_2|^2 - \left(z_1\overline{z_2} + z_2\overline{z_1}\right)}{|z_1|^2 + |z_2|^2 + \left(z_1\overline{z_2} + z_2\overline{z_1}\right)} = \frac{|z_1|^2 + |z_2|^2}{|z_1|^2 + |z_2|^2}$$

Q.39

$$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$$

$$\Rightarrow |z_1 + z_2 + z_3| = |\overline{z_1} + \overline{z_2} + \overline{z_3}| = \left|\frac{z_1\overline{z_1}}{z_1} + \frac{z_2\overline{z_2}}{z_2} + \frac{z_3\overline{z_3}}{z_3}\right|$$

$$\Rightarrow \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$$

Q.40

$$\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right|$$

Let $z_3 = \sqrt{z_1^2 - z_2^2}$ $\quad z_3^2 = z_1^2 - z_2^2$

$$\Rightarrow \left[|z_1 + z_3| + |z_1 - z_3|\right]^2 = |z_1 + z_3|^2 + |z_1 - z_3|^2 + 2|z_1 + z_3||z_1 - z_3|$$

$$\Rightarrow 2\left(|z_1|^2 + |z_3|^2\right) + 2|z_1^2 - z_3^2|$$

$$\Rightarrow 2\left(|z_1|^2 + |z_3|^2\right) + 2|z_2|^2$$

$$\Rightarrow 2\left(|z_1|^2 + |z_2|^2\right) + 2|z_1^2 - z_2^2|$$

$$\Rightarrow \left(|z_1 - z_2|^2 + |z_1 + z_2|^2\right) + 2|z_1 - z_2||z_1 + z_2|$$

$$\Rightarrow \left[|z_1 - z_2| + |z_1 + z_2|\right]^2$$

Q.41 [C]

$$\left|\frac{z-4}{z-8}\right|=1$$

Locus of Z will be x = 6

$$\Rightarrow \therefore z=6+iy$$

$$\Rightarrow \left|\frac{z-12}{z-8i}\right|=\frac{5}{3}$$

$$\Rightarrow 3|-6+iy|=5|6+i(y-8)|$$

$$\Rightarrow 9\left(36+y^2\right)=25\left(36+(y-8)^2\right)$$

$$\Rightarrow y^2-25y+136=0$$

$\Rightarrow$ y = 8, 17

$$\Rightarrow \therefore z=6+8i, 6+i17$$

Q.42 [A]

$$|z-4|<|z-2|$$

$$\Rightarrow |z-4|^2<|z-2|^2$$

$$\Rightarrow |z|^2-4\left(\bar{z}+z\right)+16<|z|^2-2\left(z+\bar{z}\right)+4$$

$$\Rightarrow 2\left(z+\bar{z}\right)>12$$

$$\Rightarrow 2(2x)>12$$

$\Rightarrow$ x > 3

$\Rightarrow$ Re (z) > 3

Q.43 [B]

$$z=1+i\tan\alpha$$

$$\Rightarrow \pi<\alpha<\frac{3\pi}{2}$$

$\Rightarrow |z| = \sqrt{1+\tan^2\alpha} = |\sec\alpha|$

$\Rightarrow |z| = -\sec\alpha$ (3rd quadrant)

Q.44 [A, B]

$$\Rightarrow \left|\frac{|\bar{z}|^2}{z\bar{z}}\right| = \left|\frac{|z|^2}{|z|^2}\right| = 1 = \left|\frac{\bar{z}}{z}\right|$$

Q.45 [C]

$$|z_1 + z_2| = \left|\frac{1}{z_1} + \frac{1}{z_2}\right|$$

$$\Rightarrow |z_1 + z_2| = \left|\frac{z_1 + z_2}{z_1 z_2}\right|$$

$\Rightarrow |z_1 z_2| = 1$

Q.46 [D]

$z^2 + |z|^2 = 0$

$\Rightarrow z^2 = -|z|^2$

$\Rightarrow z = i|z|$

$\Rightarrow$ Real (z) = 0

$\Rightarrow \therefore z = iy$

So infinite solution.

Q.47

$\Rightarrow |z| = \max\{|z-2|, |z+2|\}$

Q.48 [C]

$z = -1 + i\sqrt{3}$ z lies in 2nd quadrant

$$\Rightarrow \therefore \arg(z) = \tan^{-1}\left(-\sqrt{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Q.49

$z = -1 - i\sqrt{3}$; z lies in 3rd quadrant

$$\Rightarrow \arg(z) = \pi + \tan^{-1}\left(\sqrt{3}\right) = \frac{4\pi}{3}$$

Q.50 [A]

$$z = \frac{1 + i\sqrt{3}}{\sqrt{3} + i} \times \frac{\sqrt{3} - i}{\sqrt{3} - i} = \frac{\sqrt{3} + \sqrt{3} + 3i - i}{4}$$

$$\Rightarrow z = \frac{\sqrt{3} + i}{2}$$

$$\Rightarrow \arg(z) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

Q.51 Repeated Q.50

Q.52

$$z = \frac{13 - 5i}{4 - 9i}$$

$\Rightarrow$ arg (z) = arg (13 – 5i) – arg (4 – 9i)

$$\Rightarrow \tan^{-1}\left(\frac{-5}{13}\right) - \tan^{-1}\left(\frac{-9}{4}\right)$$

$$\Rightarrow \left[-\tan^{-1}\frac{5}{13}\right] - \left[-\tan^{-1}\frac{9}{4}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{9}{4} - \frac{5}{13}}{1 + \frac{9}{4} \times \frac{5}{13}}\right] = \tan^{-1}\left(\frac{97}{97}\right) = \frac{\pi}{4}$$

Q.53 [C]

$$z = \frac{1-i\sqrt{3}}{1+i\sqrt{3}}$$

$$\Rightarrow \arg(z) = \arg\left(1-i\sqrt{3}\right) - \arg\left(1+i\sqrt{3}\right)$$

$$\Rightarrow \left(-\frac{\pi}{3}\right) - \left(\frac{\pi}{3}\right) = \frac{-2\pi}{3} = \frac{4\pi}{3}$$

Q.54 [D]

$$z = 1 - \cos\alpha + i\sin\alpha$$

$$\Rightarrow \text{amp}(z) = \tan^{-1}\left(\frac{\sin\alpha}{1-\cos\alpha}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}}\right)$$

$$\Rightarrow \tan^{-1}\left(\cot\frac{\alpha}{2}\right) = \frac{\pi}{2} - \frac{\alpha}{2}$$

Q.55 [B]

$$z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = 1 \cdot e^{\frac{i\pi}{6}}$$

$$\Rightarrow |z| = 1, \arg(z) = \frac{\pi}{6}$$

Q.56 [B]

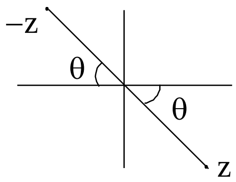
$$\arg(z) = \theta$$

$$\Rightarrow \arg(\bar{z}) = -\theta$$

Q.57

arg (z) < 0

Let arg (z) = $-\theta$; $\theta > 0$



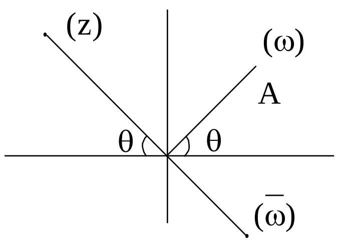
then arg (– z) – arg (z)

$(\pi-\theta)-(-\theta)=\pi$

Q.58 [D]

$|z|=|\omega|$

$\Rightarrow \arg(z)+\arg(\omega)=\pi$



$\Rightarrow \therefore z=-\bar{\omega}$

Q.59

Re(z) < 0

$\Rightarrow$ Im (z) = 0

$\Rightarrow \arg(z)\pi$

Q.60 [D]

if $\arg(z)=\theta$

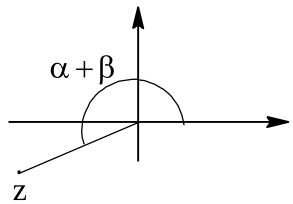
$\Rightarrow$ then $\arg(\bar{z})=-\theta$

Q.61 [C]

$\arg(z_1)=\alpha$

$\Rightarrow \arg(z_2)=\beta$

given $\alpha+\beta>\pi$



$\Rightarrow z_1 z_2 = r_1 r_2 e^{i(\alpha+\beta)}$

Principal argument $\arg(z_1 \times z_2) = -(2\pi - \alpha + \beta)$

$\Rightarrow \alpha+\beta-2\pi$

Q.62 [B]

z = – 1

$$\Rightarrow \arg\left(z^{\frac{2}{3}}\right) = \frac{2}{3}\arg(z) = \frac{2}{3}\arg(-1) = \frac{2}{3}\pi$$

Q.63 [B]

z = x + iy

$\Rightarrow$ arg (z – 1) = arg (z + 3i)

$$\Rightarrow \tan^{-1}\frac{y}{x-1} = \tan^{-1}\frac{y+3}{x}$$

$$\Rightarrow \therefore \frac{y}{x-1} = \frac{y+3}{x}$$

$\Rightarrow$ xy = xy + 3x – y – 3

$$\Rightarrow \frac{x-1}{y} = \frac{1}{3}$$

Q.64 [C]

$(1+i)^n + (1-i)^n$

$$\Rightarrow \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^n+\left[\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)\right]^n$$

$$\Rightarrow 2^{\frac{n}{2}}\left[2\cos^n\frac{\pi}{4}\right]$$

$$\Rightarrow \left(\sqrt{2}\right)^{n+2}\cos\left(\frac{n\pi}{4}\right)$$

Q.65 [A]

$$y=\cos\theta+i\sin\theta$$

$$\Rightarrow \frac{1}{y}=\bar{y}=\cos\theta-i\sin\theta$$

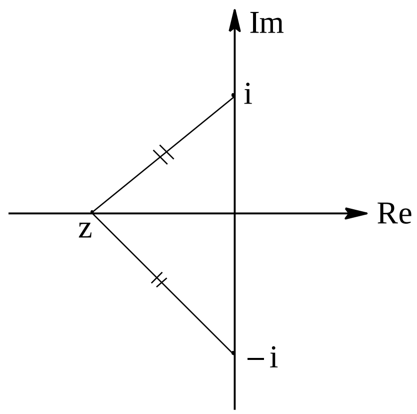
$$\Rightarrow y+\frac{1}{y}=2\cos\theta$$

Complex Number

Exercise – 1(B)

Q.1 [B]

$$w = \frac{1 - iz}{z - i} = \frac{-i(z + i)}{(z - i)}$$



$$|w| = \left|\frac{z + i}{z - i}\right| = 1$$

$$|z + i| = |z - i|$$

z lies on real axis.

Q.2 [C]

$$|z| = |z_1 - z_2| = |-3 - i| = 5$$

Q.3 [B]

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0;\ b \in R$$

radius of the circle $= |a|^2 - b > 0$

$$\therefore |a|^2 > b$$

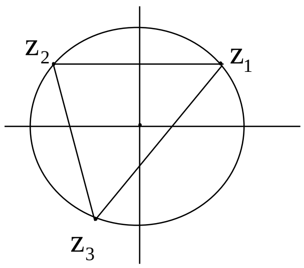
Q.4

$|z_1| = |z_2| = |z_3| = r$ (let's take)

Let $z_1 = re^{i\theta}$

$z_2 = re^{i\left(\theta + \frac{2\pi}{3}\right)}$

$z_3 = re^{i\left(\theta - \frac{2\pi}{3}\right)}$



$\therefore z_1 + z_2 + z_3 = r(0) = 0$

Q.5 [D]

$G\left(\frac{z_1 + z_2 + z_3}{3}\right) \qquad A(z_1)$

$$\therefore \text{ mid point of AG } z = \frac{\frac{z_1 + z_2 + z_3}{3} + z_1}{2} = 0$$

$\therefore 4z_1 + z_2 + z_3 = 0$

Q.6 [B]

$|z_1| = 12, \ |z_2 - 3 - 4i| = 5$

$|z_1 - z_2| = |z_1 + (-z_2 + 3 + 4i) + (-3 - 4i)|$

$||z_1| - |z_2 - 3 - 4i|| \le |z_1 + (-z_2 + 3 + 4i)| \le |z_1 - z_2| + |3 + 4i|$

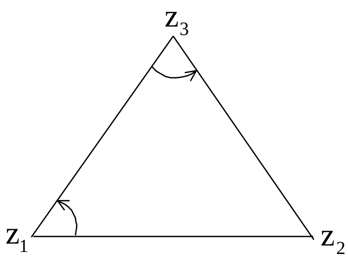
$7 \le |z_1 - z_2| + 5$

$\therefore |z_1 - z_2| \ge 2$

$|z_1 - z_2|_{min} = 2$

Q.7 [B]

For



$$\frac{z_3 - z_1}{z_2 - z_1} = e^{\frac{i\pi}{3}} = \frac{z_2 - z_3}{z_1 - z_3}$$

$$-(z_1 - z_3)^2 = (z_2 - z_3)(z_2 - z_1)$$

$$-(z_1^2 + z_3^2 - 2z_1z_3) = z_2^2 - z_1z_2 - z_2z_3 + z_1z_3$$

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Q.8 [C]

Let $\arg(z) = \theta$

Then $\arg(-iz) = \arg(-i) + \arg(z) = \frac{-\pi}{2} + \theta$

$$\therefore \arg(z) - \arg(-iz) = \frac{\pi}{2}$$

Q.9 [C]

$$\text{Re}\left(\frac{z+4}{2z-i}\right) = \frac{1}{2}$$

$$\frac{z+4}{2z-i} + \frac{\bar{z}+4}{2\bar{z}+i} = 1$$

$$(z+4)(2\bar{z}+i) + (\bar{z}+4)(2z-i) = (2z-i)(2\bar{z}+i)$$

$$2|z|^2 + iz + 8\bar{z} + 4i + 2|z|^2 - i\bar{z} + 8z - 4i = 4|z|^2 + 2zi - 2i\bar{z} + 1$$

$$zi - i\bar{z} - 8\bar{z} - 8z + 4i + 1 = 0$$

$$z(i-8) - \bar{z}(8+i) + 4i + 1 = 0$$

$$z(8-i) + \bar{z}(8+i) - 4i - 1 = 0$$

This is equation of straight line

Q.10 [C]

$$\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$$

$$z_1^2 + z_2^2 = z_1 z_2$$

For equilateral triangle with vertices z_1, z_2, z_3

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

if $z_3 = 0$

$$z_1^2 + z_2^2 = z_1 z_2$$

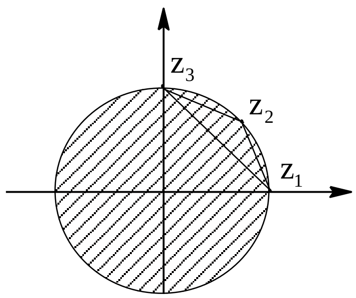
Q.11

$$z_1 = 1, z_2 = \frac{1+i}{\sqrt{2}}, z_3 = i$$

$$|z_1 - z_3| = \sqrt{2}$$

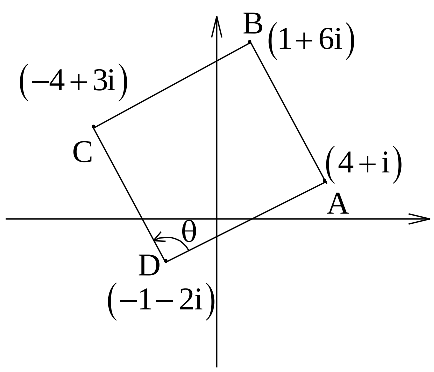
$$|z_2 - z_3| = \left|\frac{1+i}{\sqrt{2}} - i\right| = \left|\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}} - 1\right)\right| = \sqrt{\frac{1}{2} + \frac{1}{2} + 1 - \sqrt{2}} = \sqrt{2 - \sqrt{2}}$$

$$|z_1 - z_2| = \sqrt{\left(1 - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} = \sqrt{2 - \sqrt{2}}$$



Triangle is isoleces triangle

Q.12 [B]



$$|AB| = |BC| = |CD| = |DA|$$

$$\frac{Z_C - Z_D}{Z_A - Z_D} = \frac{-3+5i}{5+3i} = i = e^{\frac{i\pi}{2}}$$

$$\therefore \theta = \frac{\pi}{2}$$

$\therefore$ ABCD is square

Q.13 [B]

90^o

Q.14 [C]

By triangle inequality

$$\left||z_1| - |z_2|\right| \le |z_1 + z_2| \le |z_1| + |z_2|$$

Q.15

Q.16

$(z_1 - z_2) = \lambda (z_2 - z_3)$ for collinear

$$(3-2i) = \lambda\left(-2 + i\left(3 - \frac{a}{3}\right)\right)$$

$$3 = -2\lambda$$

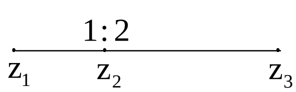
$$-2 = \lambda\left(3 - \frac{a}{3}\right)$$

$$-2 = \frac{-3}{2}\left(3 - \frac{a}{3}\right)$$

a = 5

Q.17 [B]

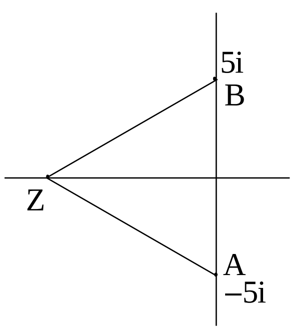
$$2z_1 - 3z_2 + z_3 = 0$$



$$z_2 = \frac{2z_1 + z_3}{2+1}$$

Collinear points.

Q.18 [A]



$$|Z - 5i| = |Z + 5i|$$

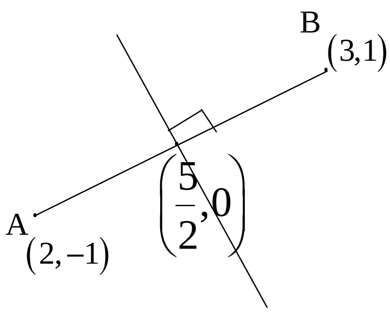
So locus of z will perpendicular bisector of AB or z lies on real axis.

$\therefore x = 0$

Q.19 [A]

$$|Z-(2-i)| = |Z-(3+i)|$$

Locus of z will be perpendicular bisector of line segment AB. A(2, – 1), B (3, 1)



$$m_{AB} = \frac{1+1}{1} = 2$$

$\therefore$ locus of z is

$$y-0 = -\frac{1}{2}\left(x-\frac{5}{2}\right)$$

$$x+2y = \frac{5}{2}$$

Q.20 [D]

$$(Z-2-3i) = \frac{\pi}{4}$$

$$\text{amp}(x-2)+i(y-3) = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{y-3}{x-2}\right) = \frac{\pi}{4}$$

y – 3 = x – 2

x – y + 1 = 0

Q.21

$$\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{6}$$

$$\arg(z-2)-\arg(z+2)=\frac{\pi}{6}$$

$$\tan^{-1}\left(\frac{y}{x-2}\right)-\tan^{-1}\left(\frac{y}{x+2}\right)=\frac{\pi}{6}$$

$$\tan^{-1}\left[\frac{\left(\frac{y}{x-2}-\frac{y}{x+2}\right)}{1+\left(\frac{y^2}{x^2-4}\right)}\right]=\frac{\pi}{6}$$

$$\frac{yx+2y-xy+2y}{x^2-4+y^2}=\sqrt{3}$$

$$x^2+y^2=4+\frac{4y}{\sqrt{3}}$$

$$x^2+y^2-\frac{4y}{\sqrt{3}}-4=0$$

$$z=\lambda+3+i\sqrt{5-\lambda^2}$$

It is a circle.

Q.22

$$x=\lambda+3 \ \& \ y=\sqrt{5-\lambda^2}$$

$$x=\lambda+3 \& y=\sqrt{5-\lambda^2}$$

$$y^2=5-(x-3)^2$$

$$(x-3)^2+y^2=5$$

Q.23 Repeated Question (21) of Ex. (2)

Q.24 [D]

$$|z-2|=2|z-3|$$

$$(x-2)^2+y^2=4\left((x-3)^2+\left(y^2\right)\right)$$

$$3x^2+3y^2-20x+32=0$$

$$x^2+y^2-\frac{20}{3}x+\frac{32}{3}=0$$

$$r=\sqrt{\left(\frac{10}{3}\right)^2-\frac{32}{3}}=\sqrt{\frac{4}{9}}=\frac{2}{3}$$

Q.25 [A]

$$i^{\frac{1}{3}}=\left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{3}}=e^{i\frac{\pi}{6}}$$

$$\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$$

$$\frac{\sqrt{3}+i}{2}$$

Q.26 [D]

$$\left(-1+i\sqrt{3}\right)^{20}=\left[2\left(e^{i\frac{2\pi}{3}}\right)\right]^{20}$$

$$2^{20}e^{i\frac{40\pi}{3}}$$

$$1^{20}e^{i\frac{4\pi}{3}}=2^{20}\left(\frac{-1-i\sqrt{3}}{2}\right)$$

Q.27 [A]

$$z=\frac{\sqrt{3}+i}{2}=-i\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$z=-i\omega$$

$$z^{69}=(-i)^{69}\,\omega^{69}=-i$$

Q.28 [C]

$$(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$$

$$\cos\left(\frac{n\pi}{2}-n\theta\right) + i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

Q.29

$$z = \left(\frac{\cos\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} = (\cos\pi + i\sin\pi)^{\frac{1}{4}}$$

$$(-1)^{\frac{1}{4}}$$

$$z^4 = -1$$

$$z = \left(\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)\right)$$

$$z_1 = e^{i\frac{\pi}{4}}$$

$$z_2 = e^{i\frac{3\pi}{4}}$$

$$z_3 = e^{i\frac{5\pi}{4}}$$

$$z_3 = e^{i\frac{7\pi}{4}}$$

Product of roots $= e^{i\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)}$

$$e^{i(4\pi)} = 1$$

Q.30 [D]

$$\left(\frac{1+\cos\theta+i\sin\theta}{i+\sin\theta+i\cos\theta}\right)^n = \cos n\theta + i\sin n\theta$$

$$\left[\frac{2\cos\frac{\theta}{2}\left(e^{i\frac{\theta}{2}}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right)}\right]^n$$

$$\left[\frac{e^{i\frac{\theta}{2}}}{ie^{i\frac{\theta}{2}}}\right]^n=\left[-ie^{i\theta}\right]^n$$

$$(-i)^n(\cos n\theta+i\sin\theta)$$

So, n = 4

Q.31 [C]

$$\left[\frac{1-\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}}{1-\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}}\right]^{10}=\left[\frac{2\sin\frac{\pi}{10}\left(\sin\frac{\pi}{20}+icoc\frac{\pi}{20}\right)}{\left(2\sin\frac{\pi}{20}\right)\left(\sin\frac{\pi}{20}-i\cos\frac{\pi}{20}\right)}\right]^{20}$$

$$\left[\frac{i\,e^{-i\frac{\pi}{20}}}{-ie^{i\frac{\pi}{20}}}\right]^{20}=\left[-e^{-i\frac{\pi}{10}}\right]^{20}=e^{-i2\pi}$$

1

Q.32 [D]

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\left(\frac{2\pi k}{7}\right)\right)$$

$$\sum_{k=1}^{6}-ie^{i\frac{2\pi k}{7}} \qquad \text{Let } \alpha=e^{i\frac{2\pi}{7}}$$

$$-i\left[\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6\right]$$

– i [–1] = i

Q.33 [B]

$$\cos\left(\theta+\frac{\theta}{2}+\frac{\theta}{2^2}+\frac{\theta}{2^3}+......\right)+\sin\left(\theta+\frac{\theta}{2}+\frac{\theta}{2^2}+\frac{\theta}{2^3}+.......\right)$$

$$\cos(2\theta)+i\sin 2\theta$$

Q.34 [C]

$$z^3=-1$$

$$z=-1,-\omega,-\omega^2$$

where ω is cube root of unity.

$$z_1z_2z_3=(-1)(-\omega)(-\omega^2)=-1$$

Q.35

$$\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)=\left[\frac{2\left(e^{i\frac{\pi}{3}}\right)}{2e^{-i\frac{\pi}{3}}}\right]^n=\left[e^{i\frac{2\pi}{3}}\right]^n$$

$$e^{i\frac{2\pi n}{3}}$$

it will be integer if n = 3.

Q.36 [C]

$$(1+\omega)^7=A+B\omega$$

$$(-\omega^2)^7=A+B\omega$$

$$(-\omega^2)=A+B\omega$$

$$1+\omega=A+B\omega$$

A = 1 = B

Q.37 [C]

$$(3t\omega+3\omega^2)^4=(-3\omega+\omega)^4=(-2\omega)^4=16\omega$$

Q.38 [A]

$$\left(3+\omega^2+\omega^4\right)^6=\left(3+\omega^2+\omega\right)^6=\left(2+0\right)^6=64$$

Q.39 [C]

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1+\omega+\omega^2=0$$

Q.40 [B]

Given question is wrong. Actual question is

$$\left(z+1\right)^3=8\left(z-1\right)^3$$

$$\left(\frac{z+1}{z-1}\right)=\left(8\right)^{\frac{1}{3}}$$

$$\frac{z+1}{z-1}=2,2\omega,2\omega^2$$

$$z=3,\frac{2\omega+1}{2\omega-1},\frac{2\omega^2+1}{2\omega^2-1}$$

$$\therefore z_1+z_2+z_3=\frac{27}{7}$$

$$\mathrm{Re}\left(z_1+z_2+z_3\right)=\frac{27}{7}$$

COMPLEX NUMBERS

Ex. 1(C)

Q.1 (c)

$x = 9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\infty \text{ terms}} \Rightarrow x = 9^{\frac{1/3}{1-1/3}}$ or $x = 3.$

$y = 4^{\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\ldots\infty \text{ terms}} \Rightarrow y = 4^{\frac{1/3}{1+1/3}}$ or $y = \sqrt{2}.$

$z = \sum_{r=1}^{\infty} \frac{1}{(1+i)^r} \Rightarrow z = \frac{1/(1+i)}{1-1/(1+i)}$ or $z = -i.$

Now $x + yz = 3 - \sqrt{2}i \Rightarrow \arg(x + yz) = -\tan^{-1}\frac{\sqrt{2}}{3}.$

Q.2 (c)

$\overline{Z} + i\overline{W} = 0 \Rightarrow Z - iW = 0$ or $\frac{Z}{W} = i.$

Now $\arg\left(\frac{Z}{W}\right) = \frac{\pi}{2} \Rightarrow \arg(ZW) + \arg\left(\frac{Z}{W}\right) = \frac{3\pi}{2}$

$\Rightarrow \arg(Z^2) = \frac{3\pi}{2}$ or $\arg(Z) = \frac{3\pi}{4}.$

Q.3 (d)

Let $P(x) = x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$

or $P(x) = x^6 - x^3 + (4x^3 - x^2 + 1)(x^2 + x + 1).$

Now for $x = \omega$ & ω^2, $x^6 - x^3 = x^2 + x + 1 = 0$, hence $P(\omega) = 0 = P(\omega^2).$

$P(x)$ is divisible be $(x - \omega)(x - \omega^2).$

Q.4 (c)

$\cos r\theta + i\sin r\theta = e^{ir\theta} \Rightarrow (\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = \sum_{r=1}^{n} e^{ir\theta}$

Hence $e^{i\frac{n(n+1)}{2}\theta} = 1$ or $\cos\frac{n(n+1)}{2}\theta + i\sin\frac{n(n+1)}{2}\theta = 1.$

$\Rightarrow \frac{n(n+1)}{2}\theta = 2m\pi$ or $\theta = \frac{4m\pi}{n(n+1)}.$

Q.5 (c)

Let $Z=\cos\theta+i\sin\theta$.

$\left|\frac{Z}{\overline{Z}}+\frac{\overline{Z}}{Z}\right|=1\Rightarrow\left|Z^2+\overline{Z}^2\right|=1$ or $2\left|\cos 2\theta\right|=1$.

Now $\cos 2\theta=\pm\frac{1}{2}\Rightarrow\theta=\frac{n\pi}{2}\pm\frac{\pi}{6}$.

As $\theta\in(0,2\pi)$, therefore $\theta=\frac{\pi}{6},\frac{\pi}{3},\frac{2\pi}{3},\frac{5\pi}{6},\frac{7\pi}{6},\frac{4\pi}{3},\frac{5\pi}{3},\frac{11\pi}{6}$.

Q.6 (A)

$\mu^2-2\mu+2=0\Rightarrow(\mu-1)^2=-1$ or $\mu-1=\pm i$, hence $\alpha-1=i$ & $\beta-1=-i$.

Now $(x+\mu)^n=(\cot\theta+\mu-1)^n$

$$\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\frac{(\cot\theta+i)^n-(\cot\theta-i)^n}{2i}$$

$$\Rightarrow\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\frac{(\cos\theta+i\sin\theta)^n-(\cos\theta-i\sin\theta)^n}{2i\sin^n\theta}$$

or $\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\frac{\sin n\theta}{\sin^n\theta}$.

Q.7 (d)

$|Z-4|=\text{Re}(Z)\Rightarrow(x-4)^2+y^2=x^2$ or $y^2=8(x-2)$.

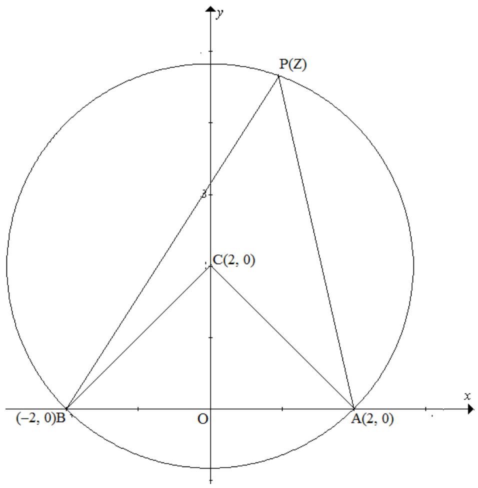
Now greatest positive $\arg(Z)$ will be greatest slope angle of tangent from origin to this parabola.

Equation of any tangent of slope m will be $y=m(x-2)+\frac{2}{m}$.

As it has to be drawn from $(0,0)$, hence $m=1$.

$\therefore$ Greatest positive $\arg(Z)=\frac{\pi}{4}$.

Q.8 (b)



As shown in figure

both A(2, 0) & B(–2, 0) lie on the circle

$|Z-2i| = 2\sqrt{2}$.

Center of the circle is C(0, 2) & radius is $2\sqrt{2}$.

Now $\angle APB = \angle ACO$.

Hence $\arg\left(\frac{Z-2}{Z+2}\right) = \frac{\pi}{4}$.

Q.9 (b)

$S = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \ldots + n\alpha^{n-1}$ ...(i)

Multiply throughout by α to get

$\alpha S = \alpha + 2\alpha^2 + 2\alpha^3 + \ldots (n-1)\alpha^{n-1} + n\alpha^n$ ...(ii)

Subtract (ii) from (i) to get

$(1-\alpha)S = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{n-1} - n\alpha^n$

Now $1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{n-1} = 0 \Rightarrow S = \frac{-n}{1-\alpha}$.

Q.10 (c)

Let $Z = a + ib$ & $\frac{2}{Z} = x + iy$, then $\frac{2(a-ib)}{a^2+b^2} = x + iy$.

As $a^2 + b^2 = 1$, thus $x = 2a$ & $y = -2b$ or $x^2 + y^2 = 4$.

Required locus is a circle of radius 2.

Q.11 (a)

$\angle AOB = \frac{\pi}{2}$ & $OA = OB \Rightarrow Z_2 = iZ_1$. Hence $\frac{Z_2}{Z_1}$ is purely imaginary.

Q.12 (c)

$\beta+\gamma=\alpha+\alpha^2+\ldots+\alpha^6=-1$ &

$$\beta\times\gamma=\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)\left(\cos\frac{6\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{12\pi}{7}\right)=2$$

hence required equation is $Z^2+Z+2=0$.

Q.13 (c)

Let $Z_1=e^{i\alpha}, Z_2=e^{i\beta}$ & $Z_3=e^{i\gamma}$.

Now $\cos\alpha+2\cos\beta+3\cos\gamma=\sin\alpha+2\sin\beta+3\sin\gamma=0\Rightarrow Z_1+2Z_2+3Z_3=0$

$\Rightarrow Z_1{}^3+8Z_2{}^3+27Z_3{}^3=18Z_1Z_2Z_3$

$\therefore \sin3\alpha+8\sin3\beta+27\sin3\gamma=18\sin(\alpha+\beta+\gamma)$.

Q.14 (A)

Let $Z=k(\cos A+i\sin A)$ & $W=k(\cos B+i\sin B)$

Now $\alpha=\dfrac{Z-\overline{W}}{k^2+Z\overline{W}}\Rightarrow\alpha=\dfrac{(\cos A-\cos B)+i(\sin A+\sin B)}{k+k(\cos A+i\sin A)(\cos B-i\sin B)}$

or $\alpha=\dfrac{(\cos A-\cos B)+i(\sin A+\sin B)}{k\{1+\cos(A-B)+i\sin(A-B)\}}$

$$\Rightarrow\alpha=\frac{\sin\frac{A+B}{2}\left(\sin\frac{B-A}{2}+i\cos\frac{A-B}{2}\right)}{k\cos\frac{A-B}{2}\left(\cos\frac{A-B}{2}+i\sin\frac{A-B}{2}\right)}$$

$$\Rightarrow\alpha=\frac{\sin\frac{A+B}{2}\left(-\sin\frac{A-B}{2}+i\cos\frac{A-B}{2}\right)}{k\cos\frac{A-B}{2}\left(\cos\frac{A-B}{2}+i\sin\frac{A-B}{2}\right)}\times\frac{\cos\frac{A-B}{2}-i\sin\frac{A-B}{2}}{\cos\frac{A-B}{2}-i\sin\frac{A-B}{2}}$$

$$\Rightarrow\alpha=\frac{i\sin\frac{A+B}{2}}{k\cos\frac{A-B}{2}}.$$

Hence Re(Z) = 0.

Q.15 (c)

$|Z^2+k|+k=|Z^2|\Rightarrow|Z^2+k|+k=|Z^2+k-k|$

Hence $\arg(Z^2)=-\arg(k)$ or $\arg(Z^2)=\pi$.

$\therefore\ \arg(Z)=\dfrac{\pi}{2}$.

Q.16 (b)

Let $f(Z) = (Z^2+1)Q(Z) + aZ + b,$

where $Q(Z)$ is the qoutient when $f(Z)$ is divided by Z^2+1

Now $f(i) = i$ & $f(-i) = 1+i$, hence

$ai + b = i$ & $-ai + b = 1 + i.$

Solving these equations simultaneously gives

$b = \frac{1+2i}{2}$ & $a = \frac{i}{2}.$

$\therefore$ remainder when $f(Z)$ is divided by Z^2+1 is $\frac{1+2i}{2} + \frac{iZ}{2}.$

Q.17 (a)

$a|Z_1| = b|Z_2| \Rightarrow \left|\frac{Z_1}{Z_2}\right| = \frac{b}{a}$ or $\frac{a\,Z_1}{b\,Z_2} = e^{i\theta}$ & $\frac{a\,Z_2}{b\,Z_1} = e^{-i\theta}.$

Hence $\frac{a\,Z_1}{b\,Z_2} + \frac{a\,Z_2}{b\,Z_1} = 2\cos\theta.$

$\left(\frac{a\,Z_1}{b\,Z_2}, \frac{a\,Z_2}{b\,Z_1}\right)$ lies on real axis between $(-2,0)$ & $(2,0)$.

Q.18 (c)

For n^{th} roots of unity

$1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{n-1} = 0$

Also let $S = 1 + 2\omega + 3\omega^2 + \ldots + n\omega^{n-1}$, then

$\omega S = \quad \omega + 2\omega^2 + \ldots + (n-1)\omega^{n-1} + n\omega^n$

From the above two relations we get $S = \frac{n}{\omega - 1}$

Now $\sum_{r=1}^{n}(ar+b)\omega^{r-1} = a\sum_{r=1}^{n} r\,\omega^{r-1} + b\sum_{r=1}^{n}\omega^{r-1}$

Or $\sum_{r=1}^{n}(ar+b)\omega^{r-1} = \frac{an}{\omega - 1}.$

Q.19 (b)

$(Z+ab)^3 = a^3 \Rightarrow Z = a - ab,\ a\omega - ab$ & $a\omega^2 - ab.$

Now side length $|(a-ab) - (a\omega - ab)| = |a(1-\omega)|$ i.e. $\sqrt{3}|a|.$

Q.20 (d)

$\left|\omega Z - 1 - \omega^2\right| = a \Rightarrow |Z+1| = a.$

Given $|Z+1| = a$ & $|Z-1| \le 2.$

Now $\left||Z+1| - 2\right| \le |Z-1| \Rightarrow |a-2| \le 2$ or $0 \le a \le 4.$

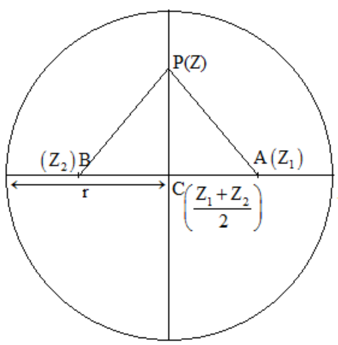
Q.21 (a)

$\left|Z^2 + 2Z\cos\alpha\right| \le |Z|^2 + 2|Z||\cos\alpha|$

Now $|Z| < \sqrt{2} - 1$ & $\cos\alpha \le 1$, hence $\left|Z^2 + 2Z\cos\alpha\right| < \left(\sqrt{2}-1\right)^2 + 2\left(\sqrt{2}-1\right).$

Or $\left|Z^2 + 2Z\cos\alpha\right| < 1.$

Q.22 (b)



Consider a circle having center at $C\left(\dfrac{Z_1+Z_2}{2}\right)$ and radius r.

Now $A(Z_1)$ & $B(Z_2)$ will be two points on a diameter such that AC = BC.

Also P(Z) will be a point on the perpendicular diameter as given PA = PB.

Clearly area will be maximum when CP = r.

Hence max. area = $\dfrac{1}{2}|Z_1 - Z_2|r$.

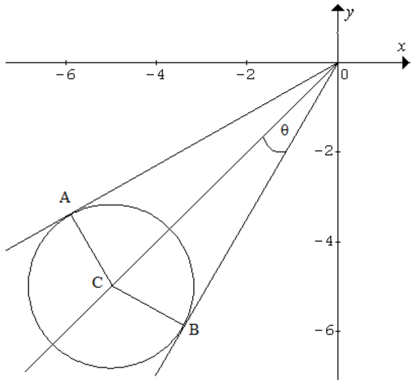
Q.23 (d)

$|Z_2 + iZ_1| = |Z_2| + |iZ_1| \Rightarrow \arg(Z_2) = \arg(iZ_1)$ or $\arg(Z_2) - \arg(Z_1) = \dfrac{\pi}{2}.$

Let $Z_1 = 3$ & $Z_2 = 4i$, then $\dfrac{Z_2 - iZ_1}{1-i} = \dfrac{i(1+i)}{2}$ or $\dfrac{-1+i}{2}$

$$\text{Area} = \frac{1}{2} \times \left\| \begin{matrix} 1 & 3 & 0 \\ 1 & 0 & 4 \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{matrix} \right\| = \frac{25}{4}.$$

Q.24 (a)



$|Z+5+5i| \le \frac{5\sqrt{3}-5}{2}$ represents a circle with center at $(-5, -5)$ and radius $\frac{5\sqrt{3}-5}{2}$.

Now OC = $5\sqrt{2}$ & BC = $\frac{5\sqrt{3}-5}{2}$, thus

$\sin\theta = \frac{\sqrt{3}-1}{2\sqrt{2}}$ or $\theta = \frac{\pi}{12}$.

Now angle made by OC with positive real axis is $\frac{5\pi}{4}$, therefore angle made by OB & OC with positive real axis are $\frac{4\pi}{3}$ & $\frac{7\pi}{6}$.

Hence least arg(Z) = $-\frac{5\pi}{6}$.

Q.25 (a)

$$\text{Case I}: |Z-1| < |Z+1| \,\&\, |Z| = |Z-1|$$
$$\text{Case II}: |Z-1| > |Z+1| \,\&\, |Z| = |Z+1|$$

$$\Rightarrow x > 0, \text{ then } x = \frac{1}{2} \,\&\, x < 0, \text{ then } x = -\frac{1}{2}.$$

Now $Z + \overline{Z} = 2\,\text{Re}(Z)$, thus $Z + \overline{Z} = 1$ or -1.

Q.26 (b)

$$\arg\left(\frac{Z_1 - \frac{Z}{|Z|}}{\frac{Z}{|Z|}}\right) = \frac{\pi}{2} \Rightarrow \frac{Z_1 - \frac{Z}{|Z|}}{\frac{Z}{|Z|}} = \left|Z_1 - \frac{Z}{|Z|}\right| e^{-\frac{\pi}{2}i}$$

$$\Rightarrow Z_1 - \frac{Z}{|Z|} = 3i\frac{Z}{|Z|} \text{ or } Z_1 = (3i+1)\frac{Z}{|Z|}.$$

Hence $|Z_1| = \sqrt{10}$.

Q.27 (b)

The required complex vector will be $\frac{\lambda}{2}\left(\frac{Z_1}{|Z_1|} + \frac{Z_2}{|Z_2|}\right)$ i.e. $\frac{\lambda}{2}\left(\frac{3+\sqrt{3}i}{2\sqrt{3}} + \frac{2\sqrt{3}+6i}{4\sqrt{3}}\right)$.

Hence any complex number of form $\mu(1+i)$ will lie along the angle bisector.

Q.28 (a)

$$|Z-2+2i| \le \big||Z|-|2-2i|\big| \Rightarrow -1 \le |Z| - 2\sqrt{2} \le 1.$$

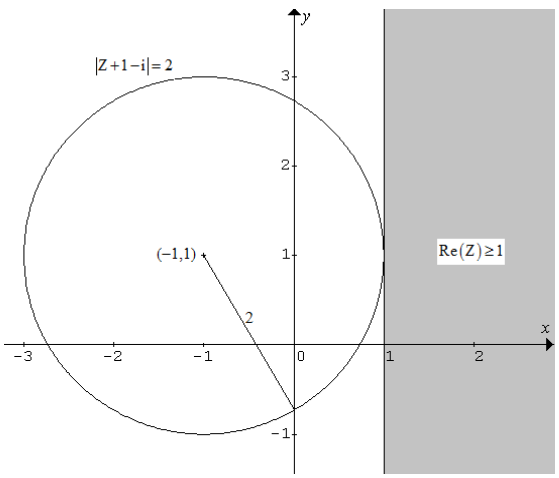
Hence least value of $|Z|$ is $2\sqrt{2}-1$.

Also arg(Z) = arg(2 − 2i).

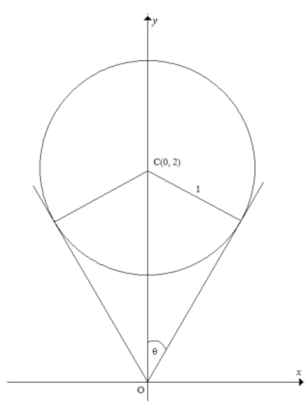
$$\therefore Z = \frac{2\sqrt{2}-1}{\sqrt{2}}(1-i).$$

Q.29 (b)

Refer the adjoining figure.



Q.30 (a)

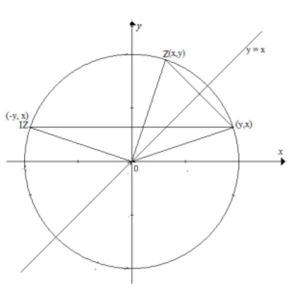


As shown in figure range of arg(Z) will be

from $\frac{\pi}{2}-\theta$ to $\frac{\pi}{2}+\theta$, where $\sin\theta = \frac{1}{2}$ i.e. $\theta = \frac{\pi}{6}$.

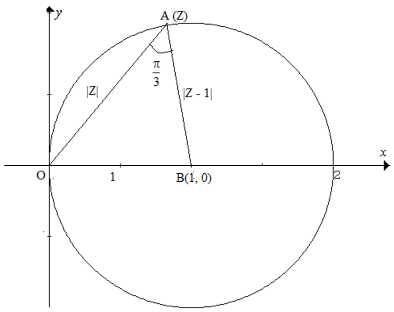
Hence $\text{Arg}(\alpha) \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$.

Q.31 (b)



Rotation of Z(x + iy) about the origin gives iZ(−y + ix).
Then reflection in Imaginary-Axis gives (y + ix),
which is equivalent to reflection of Z in the line x = y.
Hence T_1 is equivalent to composite of T_2 & T_3.

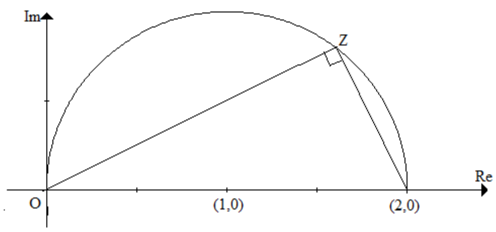
Q.32 (d)



Refer the adjoining figure.
By cosine formula,

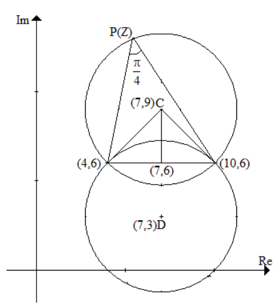
$$\cos\frac{\pi}{3}=\frac{|Z|^2+1-1}{2|Z|}\Rightarrow |Z|^2=|Z| \text{ or } |Z|=1.$$

Q.33 (c)



Refer the adjoining figure.

Q.34 (c)



Angles in same segment of a circle are equal, hence Z will move on major arc of two circles passing through (4, 6) & (10, 6) and of radius $3\sqrt{2}$ as shown in adjoining figure.

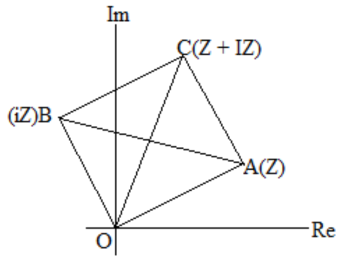
Q.35 (d)

$\cos\frac{2k\pi}{11}+i\sin\frac{2k\pi}{12}$ is 12th root of unity for $k=0,1,2,\ldots,11$.

Now $\sum_{k=0}^{11}\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)=0$ & $\sum_{k=0}^{11}\left(\sin\frac{2k\pi}{n}-i\cos\frac{2k\pi}{n}\right)=-i\sum_{k=0}^{11}\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)$,

hence $\sum_{k=1}^{11}\left(\sin\frac{2k\pi}{n}-i\cos\frac{2k\pi}{n}\right)=i.$

Q.36 (d)



$A(Z)\,\&\,B(iZ)$ are such that $OA \perp OB$.

Also $C(Z+iZ)$ will be such that OC is diagonal of Square OACB as shown in adjoining figure.

Hence required area is $\frac{1}{2}|Z|^2$.

Q.37 (c)

Let $P(Z)=(Z-1-i)(Z-1+i)Q(Z)+aZ+b$

Now $P(1+i)=3+4i \Rightarrow (1+i)a+b=3+4i \;...(i)$

& $P(1ii)=3+4i \Rightarrow (1-i)a+b=4-3i \;...(ii)$

From (i) & (ii)

$a=\left(\frac{7+i}{2}\right), b=0.$

Q.38 (a)

Note that triangle AOB is right angled isosceles triangle, hence C will be midpoint of AB.

Q.39 (b)

$(a+ib)^n=(a-ib)^n \Rightarrow e^{i(n\theta)}=e^{-i(n\theta)}$, where $\theta=\tan^{-1}\frac{b}{a}$

$\Rightarrow e^{i(2n\theta)}=1 \Rightarrow \tan^{-1}\frac{b}{a}=\frac{\pi}{n} \Rightarrow \frac{b}{a}=\tan\frac{\pi}{n}$

Clearly least positive integral value of n is 3 such that $\frac{b}{a}$ is defined and not zero.

Q.40 (a)

$|2Z_1+Z_2| \le 2|Z_1|+|Z_2| \Rightarrow |2Z_1+Z_2| \le 4.$

COMPLEX NUMBERS

Exercise – 2(A)

Q.1 (a)

$|Z_1| = |Z_2| \Rightarrow Z_1\overline{Z}_1 = Z_2\overline{Z}_2$ or $\dfrac{Z_1}{Z_2} = \dfrac{\overline{Z}_2}{\overline{Z}_1}$

$\Rightarrow \dfrac{Z_1 + Z_2}{Z_1 - Z_2} = -\dfrac{\overline{Z}_1 + \overline{Z}_2}{\overline{Z}_1 - \overline{Z}_2}$

Hence $\dfrac{Z_1 + Z_2}{Z_1 - Z_2}$ is purely imaginary.

Q.2 (b)

$|Z|^{n-2} Z^2 + |Z|^{n-2} Z - |Z|^n = 0 \Rightarrow |Z|^{n-2}\left(Z^2 + Z - |Z|^2\right) = 0,$

hence $|Z| = 0$ or $Z^2 + Z - |Z|^2 = 0$

$\Rightarrow x - 2y^2 + y(2x+1)i = 0.$

Hence $x = 2y^2$ & $y = 0$ $\left\{\text{given } x \neq -\dfrac{1}{2}\right\}$.

Therefore $x = 0, y = 0.$

Hence one solution is possible as $Z = 0$ if $n \geq 2.$

Q.3 (b)

Let the root be yi, then $Z^4 + a_1Z^3 + a_2Z^2 + a_3Z + a_4 = 0$ gives

$y^4 - a_1 i\, y^3 - a_2y^2 + a_3 i\, y + a_4 = 0 \ \ldots(i)$

Also as the coefficients are real, hence $-yi$ must also be a root.

$\therefore\ y^4 + a_1 i\, y^3 - a_2y^2 - a_3 i\, y + a_4 = 0 \ \ldots(ii)$

Subtracting (i) from (ii) we get $y^2 = \dfrac{a_3}{a_1}$

Adding (i) & (ii) we get $y^4 - a_2y^2 + a_4 = 0$

$\therefore \dfrac{a_3{}^2}{a_1{}^2} - \dfrac{a_2a_3}{a_1} + a_4 = 0$ or $\dfrac{a_3}{a_1a_2} + \dfrac{a_1a_4}{a_2a_3} = 1.$

Q.4 (b)

$$(A+1)^n = A^n \Rightarrow A+1 = Ae^{\frac{2k\pi}{n}i} \quad \therefore \quad A = \frac{1}{e^{\frac{2k\pi}{n}i}-1} \text{ or } \frac{\cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n} - 1}{2-2\cos\frac{2k\pi}{n}}$$

$$\Rightarrow A = i\frac{\cos\frac{k\pi}{n} - i\sin\frac{k\pi}{n}}{2\sin\frac{k\pi}{n}} \text{ or } A^n = i^n \frac{\cos k\pi}{2^n \sin^n \frac{k\pi}{n}}$$

Now $i^n \cos k\pi = 2^n \sin^n \frac{k\pi}{n}$, therefore least value of $n = 6$.

Q.5 (c)

Given $Z_1\left(Z_1^2 - 3Z_2^2\right) = 2$ & $Z_2\left(3Z_1^2 - Z_2^2\right) = 11$.

Now $Z_1\left(Z_1^2 - 3Z_2^2\right) + i Z_2\left(3Z_1^2 - Z_2^2\right) = 2 + 11i \Rightarrow \left(Z_1 + iZ_2\right)^3 = 2 + 11i \; ...(i)$

& $Z_1\left(Z_1^2 - 3Z_2^2\right) - i Z_2\left(3Z_1^2 - Z_2^2\right) = 2 - 11i \Rightarrow \left(Z_1 - iZ_2\right)^3 = 2 - 11i \; ...(ii)$

Further $\left(Z_1 + iZ_2\right)^3\left(Z_1 - iZ_2\right)^3 = 125$ or $Z_1^2 + Z_2^2 = 5$.

Q.6 (b)

Let $Z = x + iy$, then $|2Z - 1| = |Z - 2| \Rightarrow (2x-1)^2 + 4y^2 = (x-2)^2 + y^2$

or $x^2 + y^2 = 1$.

Now $\left|(Z_1 - \alpha) + (Z_2 - \beta) + \alpha + \beta\right| \le |Z_1 - \alpha| + |Z_2 - \beta| + |\alpha| + |\beta|$

$\Rightarrow |Z_1 + Z_2| \le 2(\alpha + \beta)$ or $\left|\frac{Z_1 + Z_2}{\alpha + \beta}\right| \le 2$

Hence $\left|\frac{Z_1 + Z_2}{\alpha + \beta}\right| \le 2|Z|$.

Q.7 (c)

$$\left|a_0 Z^n + a_1 Z^{n-1} + a_2 Z^{n-2} + \ldots + a_n\right| = 3$$

$$\Rightarrow |a_0||Z|^n + |a_1||Z|^{n-1} + |a_2||Z|^{n-2} + \ldots + |a_n| \ge 3$$

$$\Rightarrow |Z|^n + |Z|^{n-1} + |Z|^{n-2} + \ldots + 1 > \frac{3}{2} \text{ as } |a_i| < 2$$

when $|Z| \ge 1$, anyway the inequality holds so when $|Z| < 1$, then LHS will be greatest as $n \to \infty$.

Hence $\dfrac{1}{1-|Z|} > \dfrac{3}{2}$ or $|Z| > \dfrac{1}{3}$.

Q.8 (b)

$$|Z| = \left|Z + \frac{1}{Z} - \frac{1}{Z}\right| \Rightarrow |Z| \le \left|Z + \frac{1}{Z}\right| + \frac{1}{|Z|}$$

$$\therefore \left|Z + \frac{1}{Z}\right| \ge |Z| - \frac{1}{|Z|} \text{ So } \left|Z + \frac{1}{Z}\right| \ge \frac{8}{3}.$$

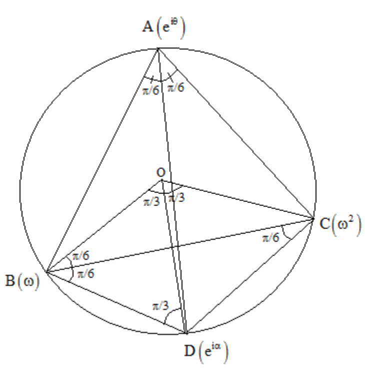
Q.9 (c)

$$|Z_r - r| \ge \left||Z_r| - r\right| \Rightarrow \left||Z_r| - r\right| \le r \text{ or } |Z_r| \le 2r.$$

Hence $\displaystyle\sum_{r=1}^{n} |Z_r| \le n(n+1)$

Also $\left|\displaystyle\sum_{r=1}^{n} Z_r\right| \le \displaystyle\sum_{r=1}^{n} |Z_r|$, therefore $\left|\displaystyle\sum_{r=1}^{n} Z_r\right| \le n(n+1)$

Q.10 (d)



Refer the adjoining figure.

$\frac{1}{2}\angle BOC = \angle BAC = \frac{\pi}{3}$ & $\angle CAD = \angle BAD = \angle CBD = \frac{\pi}{6}$

$\Rightarrow \angle BOD = \frac{\pi}{3}$

Hence D is (-1, 0).

$\therefore\ OD = \omega + \omega^2$

Q.11 (c)

$$Z = \frac{at+b}{t-c} \Rightarrow t = \frac{b+Zc}{Z-a}$$

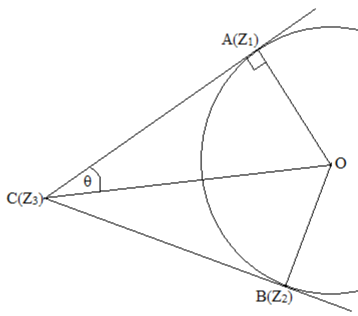
$|t| = 1 \Rightarrow |Zc+b| = |Z-a|$ or $\frac{\left|Z+\frac{b}{c}\right|}{|Z-a|} = |c|$. As $|c| \neq 1$, hence locus will be a circle.

Q.12 (a)

$$Z_k = 1 + a + a^2 + \ldots + a^{k-1} \Rightarrow Z_k = \frac{1-a^k}{1-a} \text{ or } Z_k - \frac{1}{1-a} = -\frac{a_k}{1-a}$$

$$\Rightarrow \left|Z_k - \frac{1}{1-a}\right| = \left|\frac{a_k}{1-a}\right| < \frac{1}{|1-a|}$$. Hence Z_k lies inside the circle $\left|Z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$.

Q.13 (b)



$$\frac{Z_1}{Z_2} = e^{(\pi-2\theta)i} \ \& \ \frac{Z_3 - Z_1}{Z_2 - Z_1} = e^{2i\theta} \Rightarrow \frac{Z_3 - Z_1}{Z_3 - Z_2} = -\frac{Z_1}{Z_2}$$

or $Z_3 = \frac{2Z_1Z_2}{Z_1 + Z_2}$.

Q.14 (d)

$$Z = \frac{1}{2+\cos\theta + i\sin\theta} \Rightarrow Z = \frac{1}{2+e^{i\theta}} \text{ or } e^{i\theta} = \frac{1-2Z}{Z}.$$

Hence $|2Z-1| = |Z|$. now let $Z = x+iy$, then $(2x-1)^2 + 4y^2 = x^2 + y^2$.

or $3x^2 + 3y^2 - 4x + 1 = 0$.

Hence locus of Z is a circle with center at x – axis.

Q.15 (d)

p^{th} roots of unity $= e^{\frac{2k_1\pi}{p}i}$ for $k_1 = 1, 2, 3, ..., p-1$ &

q^{th} roots of unity $= e^{\frac{2k_2\pi}{q}i}$ for $k_2 = 1, 2, 3, ..., q-1$

Now let $\frac{k_1}{p} = \frac{k_2}{q}$ for some $\{k_1, k_2\}$, then $qk_1 = pk_2$.

As p & q are distinct prime numbers hence the above conclusion is a contradiction.

Number of such {p, q} = 0.

Q.16 (d)

$$\left(\frac{Z+1}{Z}\right)^n = 1 \Rightarrow \frac{Z+1}{Z} = e^{\frac{2k\pi}{n}i} \text{ for } k = 1, 2, ..., n-1.$$

$$\text{or } Z = \frac{1}{\cos\frac{2k\pi}{n} - 1 + i\sin\frac{2k\pi}{n}} \text{ or } Z = \frac{\cos\frac{2k\pi}{n} - 1 - i\sin\frac{2k\pi}{n}}{\left(\cos\frac{2k\pi}{n} - 1\right)^2 + \sin^2\frac{2k\pi}{n}}$$

$$\Rightarrow \operatorname{Re}(Z) = \frac{\cos\frac{2k\pi}{n} - 1}{2 - 2\cos\frac{2k\pi}{n}} \text{ or } \operatorname{Re}(Z) = -\frac{1}{2}$$

Hence $\sum_{k=1}^{n-1} \operatorname{Re}(Z) = -\frac{n-1}{2}$.

Q.17 (c)

$$\left(\frac{Z+1}{Z}\right)^4 = 16 \Rightarrow \frac{Z+1}{Z} = -2, 2, -2i, 2i \text{ or } Z = -\frac{1}{3}, 1, -\frac{1-2i}{5} \,\&\, -\frac{1+2i}{5}.$$

Points representing these roots on argand plane are

$$A\left(-\frac{1}{3}, 0\right), B(1, 0), C\left(-\frac{1}{5}, \frac{2}{5}\right) \& D\left(-\frac{1}{5}, \frac{2}{5}\right).$$

Point which is equidistant from these is $\left(\frac{1}{3}, 0\right)$.

Q.18 (a)

$$(Z-1)(Z-Z_1)(Z-Z_2)\ldots(Z-Z_{n-1}) = Z^n - 1 \Rightarrow (Z-Z_1)(Z-Z_2)\ldots(Z-Z_{n-1}) = \frac{Z^n-1}{Z-1}$$

or $\ln(Z-Z_1)+\ln(Z-Z_2)+\ldots+\ln(Z-Z_{n-1}) = \ln(Z^n-1)+\ln(Z-1)$

Differentiating w.r.to Z we get

$$\frac{1}{Z-Z_1}+\frac{1}{Z-Z_2}+\ldots+\frac{1}{Z-Z_{n-1}} = \frac{n\,Z^{n-1}}{Z^n-1}+\frac{1}{Z-1}.$$

Substitute $Z=3$ gives

$$\frac{1}{3-Z_1}+\frac{1}{3-Z_2}+\ldots+\frac{1}{3-Z_{n-1}} = \frac{n\,3^{n-1}}{3^n-1}+\frac{1}{2}.$$

Q.19 [d]

$z^3 - 2z^2 + z - 1 = 0$ $\qquad\qquad\ldots\ldots\ldots \alpha,\beta,\gamma$

Let $y = \frac{1}{z-1}$

$\Rightarrow z = \frac{1+y}{y}$

By theory of equations,

$\Rightarrow \left(\frac{1+y}{y}\right)^3 - 2\left(\frac{1+y}{y}\right)^2 + \left(\frac{1+y}{y}\right) - 1 = 0$ will have the roots $\frac{1}{\alpha-1}, \frac{1}{\beta-1} \& \frac{1}{\gamma-1}$

$\Rightarrow y^3 - y - 1 = 0$ $\qquad\qquad\ldots\ldots\ldots \frac{1}{\alpha-1}, \frac{1}{\beta-1}, \frac{1}{\gamma-1}$

Sum of roots $\frac{1}{\alpha-1}+\frac{1}{\beta-1}+\frac{1}{\gamma-1} = 0$

Required value $= \frac{\alpha}{\alpha-1}+\frac{\beta}{\beta-1}+\frac{\gamma}{\gamma-1}$

$\Rightarrow \frac{\alpha-1+1}{\alpha}+\frac{\beta-1+1}{\beta-1}+\frac{\gamma-1+1}{\gamma-1}$

$\Rightarrow 3+\left(\frac{1}{\alpha-1}+\frac{1}{\beta-1}+\frac{1}{\gamma-1}\right)$

$\Rightarrow 3 + 0 = 3.$

Q.20 [b]

Point $Q(z_2)$ will be image of P (1 + i) in the line $(3-4i)z+(3+4i)\bar{z}+1=0$, if

$$\Rightarrow (3-4i)z_2+(3+4i)\left(\overline{1+i}\right)+1=0$$

$$\Rightarrow z_2=\frac{-\left[(3+4i)(1-i)+1\right]}{(3-4i)}=-\left(\frac{4+7i}{5}\right)$$

Q.21 [a]

Let z = ib

$$\Rightarrow |z-2-3i|^2+|z+4-i|^2=4|z-1+2i|^2$$

$$\Rightarrow (a-2)^2+(b-3)^2+(a+4)^2+(b-1)^2=4\left[(a-1)^2+(b+2)^2\right]$$

$$\Rightarrow a^2+b^2-6a+12b-5=0$$

$$\Rightarrow \text{radius of circle}=\sqrt{3^2+6^2+5}=5\sqrt{2}$$

Q.22 [d]

$$\begin{vmatrix}1 & \omega & \omega^2\\ 2 & 1+\omega & 1+\omega^2\\ 3 & 2+\omega & 2+\omega^2\end{vmatrix}$$

$C_3 \to C_3 - C_2$

$$\Rightarrow \begin{vmatrix}1 & \omega & \omega^2-\omega\\ 2 & 1+\omega & \omega^2-\omega\\ 3 & 2+\omega & \omega^2-\omega\end{vmatrix}$$

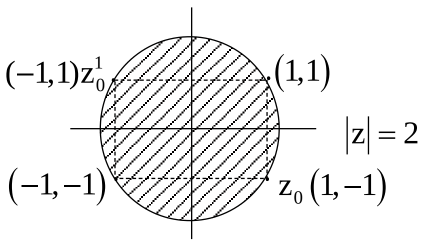
$$\Rightarrow (\omega^2-\omega)\begin{vmatrix}1 & \omega & 1\\ 2 & 1+\omega & 1\\ 3 & 2+\omega & 1\end{vmatrix}$$

$$\Rightarrow C_2 \to C_2 + C_3 - C_1$$

$$\Rightarrow \begin{vmatrix} 1 & \omega & 1 \\ 2 & \omega & 1 \\ 3 & \omega & 1 \end{vmatrix}$$

$\Rightarrow 0$

Q.23 [d]



By geometry, one can observe that greatest value of $|z-(1+i)|+|z-(-1-i)|$ will occur if $z = z_0$ or $z = z_0^1$.

Hence, maximum value will be

2 + 2 = 4

Q.24 [b]

$e^{i\theta} = \cos\theta + i\sin\theta$ ……….(i)

$\Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta$ ……….(ii)

From (i) & (ii)

$$\Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\Rightarrow \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\Rightarrow \tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{i\left(e^{i\theta}+e^{-i\theta}\right)}$$

$$\Rightarrow \tan(i\sin\theta) = \frac{\left(e^{-\sin\theta}-e^{\sin\theta}\right)}{i\left(e^{-\sin\theta}+e^{\sin\theta}\right)}$$

Hence, purely imaginary

Q.25 [a]

From Binomial Theorem, we know that

$$\Rightarrow (1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$$

Substituting x = 1,

$$\Rightarrow \quad 2^n = {}^nC_o + {}^nC_1 + {}^nC_2 + ... \qquad ....(i)$$

Substituting x = – 1

$$\Rightarrow \quad 0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - .... \qquad ....(ii)$$

Adding (i) & (ii)

$$\Rightarrow \quad 2^{n-1} = \left({}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + ...\right) \qquad ....(iii)$$

Also, $(1+ix)^n = {}^nC_0 + {}^nC_1(ix) - {}^nC_2x^2 - {}^nC_3ix^3 + {}^nC_4x^4 + {}^nC_5ix^5 + ...$

Substituting x = 1

$$\Rightarrow (1+i)^n = \left({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + ...\right) + i\left({}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + ...\right)$$

$$\Rightarrow \left(\sqrt{2}\right)^n e^{i\frac{2\pi}{4}}$$

$$\Rightarrow 2^{\frac{n}{2}}\cos\frac{n\pi}{2} + i2^{\frac{n}{2}}\sin\frac{4\pi}{4} = \left({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + ...\right) + i\left({}^nC_1 - {}^nC_3 + {}^nC_5.......\right)$$

Equating real parts of both sides,

$$\Rightarrow 2^{\frac{n}{2}}\cos\frac{n\pi}{4} = {}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 \qquad ....(iv)$$

Adding (iii) & (iv), we get

$$\Rightarrow 2^{n-1} + 2^{\frac{n}{2}}\cos\frac{n\pi}{4} = 2\left({}^nC_0 + {}^nC_4 + {}^nC_8 + {}^nC_{12} + ...\right)$$

Substituting n = 20,

$$\Rightarrow \frac{1}{2}\left(2^{19} + 2^{10}\cos 5\pi\right) = {}^{20}C_0 + {}^{20}C_4 + {}^{20}C_8 + {}^{20}C_{12} + {}^{20}C_{16} + {}^{20}C_{20}$$

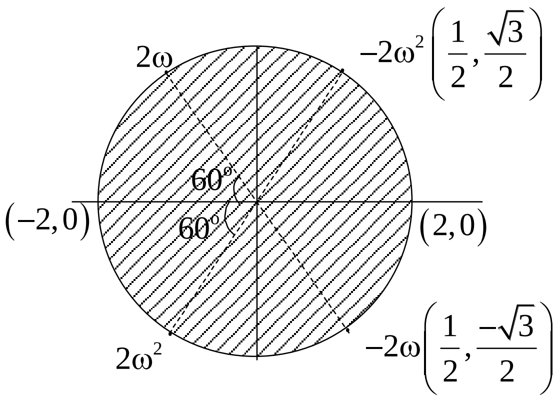
$\Rightarrow 2^9\left(2^9-1\right)$

Q.26

$z^2-2z+4=0$

$\Rightarrow z=\dfrac{2\pm\sqrt{4-16}}{2}$

$\Rightarrow 1+\sqrt{3}i$ or $1-\sqrt{3}i$



Required value =

$$\left(\frac{-2\omega}{2}+\frac{2}{-2\omega}\right)^2+\left(\omega^2+\frac{1}{\omega^2}\right)^2+\left((-\omega)^3+(-\omega^2)^3\right)^2+\left(\omega^4+\frac{1}{\omega^4}\right)^2+\left((-\omega)^5+\frac{1}{(-\omega)^5}\right)^2+\left((-\omega)^6+\frac{1}{(-\omega)^6}\right)^2+\ldots\ldots$$

$\Rightarrow\left(-\omega-\omega^2\right)^2+\left(\omega^2+\omega\right)^2+(1+1)^2+\left(\omega+\omega^2\right)^2+\left(-\omega-\omega^2\right)^2+(1+1)^2+\ldots\ldots\ldots\ldots$

$\Rightarrow (1+1+4)+(1+1+4)+\ldots\ldots..$

$\Rightarrow 6\times 8=48$

$\Rightarrow$ 8 times repetition

Q.27 [b]

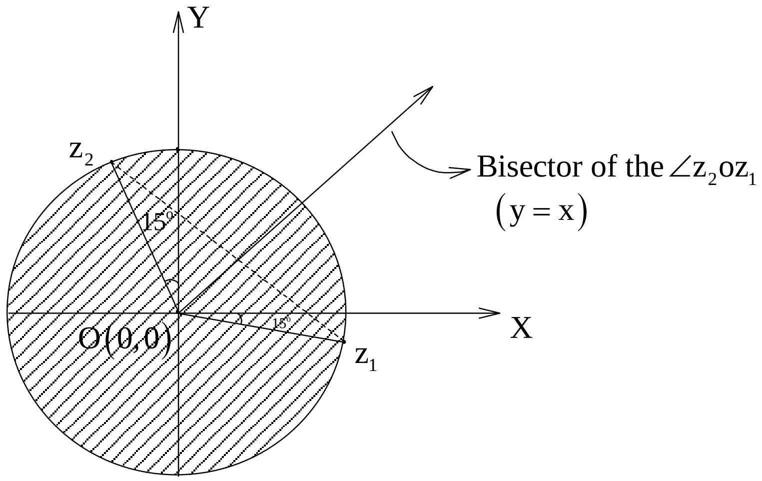
$z_1=\left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)-\left(\dfrac{\sqrt{3}-1}{2\sqrt{2}}\right)i$

$\Rightarrow \cos\left(15^\circ\right)-i\sin\left(15^\circ\right)$

$\Rightarrow \cos\left(-15^{o}\right)+i\sin\left(-15^{o}\right)$

$\Rightarrow z_2 = -\sin\left(15^{o}\right)+i\cos\left(15^{o}\right)$

$\Rightarrow \cos\left(90^{o}+15^{o}\right)+i\sin\left(90^{o}+15^{o}\right)$



Q.28 [b]

$z_n = e^{i\frac{\pi}{(2n+1)(2n+3)}}$

$\Rightarrow z_1 z_2 z_3 \ldots\ldots\ldots. z_n = e^{i\sum_{r=1}^{n}\frac{\pi}{2}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)}$

$\Rightarrow e^{i\frac{\pi}{2}\left(\frac{1}{3}-\frac{1}{2n+2}\right)}$

$\Rightarrow \underset{n\to\infty}{\text{Limit}}\left(z_1 z_2 z_3 \ldots\ldots\ldots. z_n\right)$

$\Rightarrow \underset{n\to\infty}{\text{Limit}}\left(e^{i\frac{\pi}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right)}\right) e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

Q.29 [a]

$\Delta OAP \sim \Delta OQR$

$\Rightarrow \frac{OR}{OQ} = \frac{OP}{OA}$

Using concept of Rotation

$$\Rightarrow \frac{Z-0}{|Z-0|}=\frac{Z_2-0}{|z_2-0|}e^{i\theta} \qquad \text{..........(i)}$$

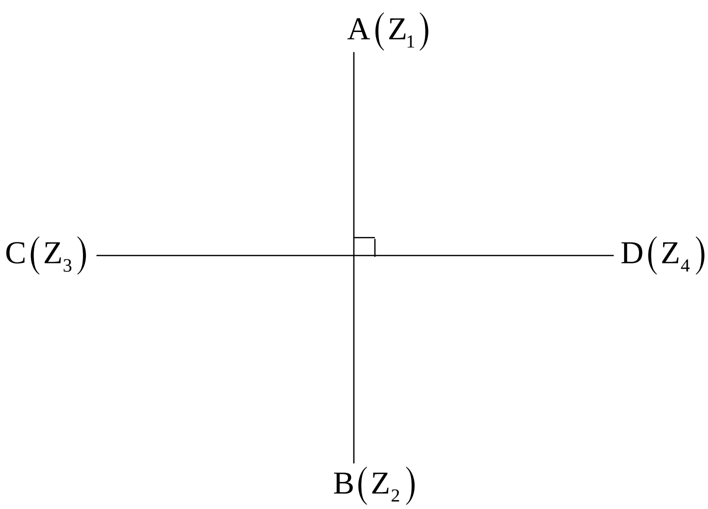
$$\Rightarrow \frac{Z_1-0}{|Z_1-0|}=\frac{1-0}{OA}e^{i\theta} \qquad \text{..........(ii)}$$

From eq^n (i) & (ii)

$$\Rightarrow \frac{Z}{OA\times OR}\times e^{i\theta}=\frac{Z_2Z_2e^{i\theta}}{OP\times OQ}$$

$$\Rightarrow Z=Z_1Z_2$$

Q.30 [a]



We know that, if $AB\perp CB$, then

$$\Rightarrow \frac{Z_1-Z_2}{Z_3-Z_4}+\frac{\overline{Z}_1-\overline{Z}_2}{\overline{Z}_3-\overline{Z}_4}=0$$

$$\Rightarrow \frac{Z_1-Z_2}{\overline{Z}_1-\overline{Z}_2}+\frac{Z_3-Z_4}{\overline{Z}_3-\overline{Z}_4}=0$$

$$\Rightarrow \omega_1+\omega_2=0$$

Solutions

COMPLEX NUMBERS

Ex. 2(B)

Q.1 (A), (C)

If vertices of an equilateral triangle are $Z_1, Z_2 \& Z_3$, then $Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$.
Now if $Z_1 = \omega \& Z_2 = \omega^2$, then $Z_3 = 1$ or -2.

Q.2 (A), (C)

Opposite sides of rhombus are parallel, hence $\frac{Z_1 - Z_4}{Z_2 - Z_3}$ is purely real.

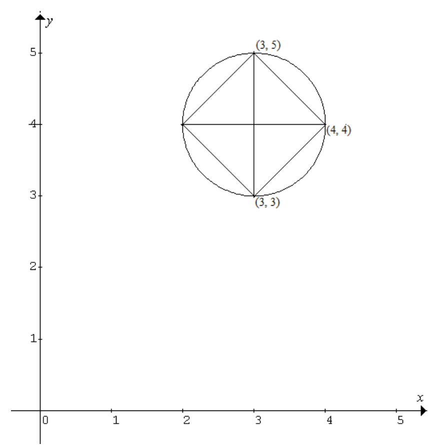
Diagonals of a rhombus are equal in length and mutually perpendicular, hence

$\frac{Z_1 - Z_3}{Z_2 - Z_4}$ is purely imaginary $\& |Z_1 - Z_3| = |Z_2 - Z_4|$.

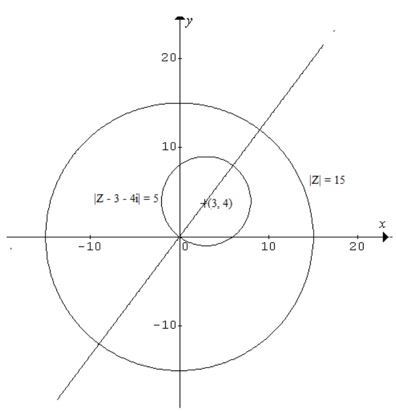
QS will bisect the angle between PS & RS, hence $\text{amp}\frac{Z_1 - Z_4}{Z_2 - Z_4} = \text{amp}\frac{Z_2 - Z_4}{Z_3 - Z_4}$.

Q.3 (B), (C)

As the rectangle is of maximum area hence it must be a square of side length $\sqrt{2}$.
Now the points on $|Z - 3 - 4i| = 1$ at a distance of $\sqrt{2}$ from (4, 4) are (3, 5) & (3, 3).



Q.4 (A), (D)



Minimum distance between any point on the bigger circle and any point on the smaller circle will be equal to radius of bigger circle – diameter of smaller circle i.e. 5. Hence
$|Z_1 - Z_2|_{min} = 5$

Maximum distance between any point on the bigger circle and any point on the smaller circle will be equal to radius of bigger circle + diameter of smaller circle i.e. 25. Hence
$|Z_1 - Z_2|_{max} = 25$

Q.5 (A), (D)

If vertices of an equilateral triangle are $Z_1, Z_2 \& Z_3$, then $Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$.

Hence $Z^2 + Z^2 + (1-Z)^2 = -Z^2 - Z(1-Z) + (1-Z)Z \Rightarrow 4Z^2 - 2Z + 1 = 0$.

$\therefore \sum Z = \frac{1}{2}$ & $\prod Z = \frac{1}{4}$.

Q.6 (B), (C)

Let locus of point P with affix Z be C(Z) and points A & B have affixes Z_1& Z_2.

$C(Z): \frac{|Z - Z_1|}{|Z - Z_2|} = 3 \Rightarrow C(Z): (Z - Z_1)(\bar{Z} - \bar{Z}_1) = 9(Z - Z_2)(\bar{Z} - \bar{Z}_2)$.

or $C(Z): 8|Z|^2 + (Z_1 - 9Z_2)\bar{Z} + (\bar{Z}_1 - 9\bar{Z}_2)Z + 9|Z_2|^2 - |Z_1|^2 = 0$.

Hence locus of P is a circle.

Now $C(Z_1) = 9\left(|Z_1|^2 - Z_2\bar{Z}_1 - \bar{Z}_2Z_1 + |Z_2|^2\right) = 9(Z_1 - Z_2)(\bar{Z}_1 - \bar{Z}_2)$ or $C(Z_1) = 9|Z_1 - Z_2|^2 > 0$.

Hence A lies outside the locus of P.

Also $C(Z_2) = -|Z_2|^2 - Z_1\bar{Z}_2 - \bar{Z}_1Z_2 - |Z_1|^2 = -(Z_1 - Z_2)(\bar{Z}_1 - \bar{Z}_2)$ or $C(Z_2) = -|Z_1 - Z_2|^2 < 0$.

Hence B lies inside the locus of P.

Q.7 (A), (D)

Let Z = x + yi, then as given $\frac{y}{x+a} = \frac{1}{\sqrt{3}}$ & $\frac{y}{x-a} = -\sqrt{3} \Rightarrow \frac{y}{x} = \sqrt{3}$.

Also $\frac{y}{x+a} = \frac{1}{\sqrt{3}}$ & $\frac{y}{x-a} = -\sqrt{3} \Rightarrow x = \frac{a}{2}$ & $y = \frac{\sqrt{3}a}{2}$, hence |Z| = a.

Q.8 (A), (C)

$-i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} \Rightarrow (-1)^{1/3} = \cos\left(\frac{2k\pi}{3} + \frac{\pi}{2}\right) + i\sin\left(\frac{2k\pi}{3} + \frac{\pi}{2}\right)$, where $k = 0,1,2$.

or $i, -\frac{\sqrt{3}}{2} - \frac{i}{2}$ & $\frac{\sqrt{3}}{2} - \frac{i}{2}$.

Q.9 (B), (C)

$\text{amp}(Z_1Z_2) = 0$ & $|Z_1Z_2| = 1 \Rightarrow Z_2Z_1 = 1$,

Also $|Z_1| = |Z_2| = 1 \Rightarrow Z_1 = \frac{1}{Z_2} = \bar{Z}_2$ or $Z_1 = \bar{Z}_2$.

Q.10 (B), (C)

$\left|Z_1^2 - Z_2^2\right| = \left|\bar{Z}_1^{\,2} + \bar{Z}_2^{\,2} - 2\bar{Z}_1\bar{Z}_2\right| \Rightarrow \left|Z_1^2 - Z_2^2\right| = \left|\bar{Z}_1 - \bar{Z}_2\right|^2$

but $|Z| = |\bar{Z}|$, hence $|Z_1 - Z_2||Z_1 + Z_2| = |Z_1 - Z_2|^2$

$\therefore\ |Z_1 + Z_2| = |Z_1 - Z_2|$ as $Z_1 \neq Z_2$.

Now $|Z_1 \pm Z_2|^2 = |Z_1|^2 + |Z_2|^2 \pm 2\left(Z_1\bar{Z}_2 + Z_2\bar{Z}_1\right)$ & $|Z_1 + Z_2| = |Z_1 - Z_2| \Rightarrow Z_1\bar{Z}_2 + Z_2\bar{Z}_1 = 0$

Or $\frac{Z_1}{Z_2} + \left(\frac{\bar{Z}_1}{\bar{Z}_2}\right) = 0$ which implies $\frac{Z_1}{Z_2}$ is purely imaginary and $\arg\left(\frac{Z_1}{Z_2}\right) = \frac{\pi}{2}$.

Q.11 (A), (C), (D)

$Z' = Z \times e^{i\alpha}$ & $Z'' = Z \times e^{-i\alpha} \Rightarrow Z' \times Z'' = Z^2$.

Hence Z', Z & Z'' are in G.P.

Clearly $Z' + Z'' = Z\left(e^{i\alpha} + e^{-i\alpha}\right)$ or $Z' + Z'' = 2Z\cos\alpha$.

Also $(Z')^2 + (Z'')^2 = Z^2\left(e^{2i\alpha} + e^{-2i\alpha}\right)$ or $(Z')^2 + (Z'')^2 = 2Z^2\cos 2\alpha$.

Q.12 (A), (B), (C)

$|Z_1 + iZ_2| \le |Z_1| + |Z_2| \Rightarrow |Z_1 + iZ_2| \le 17$.

$|Z_1 + (1+i)Z_2| \ge \big||Z_1| - |(1+i)Z_2|\big| \Rightarrow |Z_1 + (1+i)Z_2| \ge 13 - 4\sqrt{2}$.

$\left||Z_2| - \frac{4}{|Z_2|}\right| \le \left|Z_2 + \frac{4}{Z_2}\right| \le |Z_2| + \frac{4}{|Z_2|} \Rightarrow 3 \le \left|Z_2 + \frac{4}{Z_2}\right| \le 5$.

Now $\frac{|Z_1|}{\left|Z_2 + \frac{4}{Z_2}\right|} = \frac{13}{\left|Z_2 + \frac{4}{Z_2}\right|} \Rightarrow \frac{13}{5} \le \frac{|Z_1|}{\left|Z_2 + \frac{4}{Z_2}\right|} \le \frac{13}{3}$.

Q.13 (A), (B), (D)

$\left|\frac{Z - i/2}{Z + 1}\right| = \frac{m}{2}$ will be a circle for all values of m except m = 2.

For m = 2 this eq. will represent a straight line.

Q.14 (A), (D)

The equation $\arg(Z) = \frac{\pi}{6}$ represents the ray $\sqrt{3}y = x$.

The equation $|Z - 2\sqrt{3}\,i| = r$ represents the circle $x^2 + \left(y - 2\sqrt{3}\right)^2 = r^2$.

Solving the two equations simultaneously, we get $4y^2 - 4\sqrt{3}y + 12 - r^2 = 0$.

For this equation to have distinct real roots $48-16\left(12-r^2\right)>0$ or $r^2>9$.

Hence $r>3$ & $[r]\neq 2$.

Q.15 (B), (C)

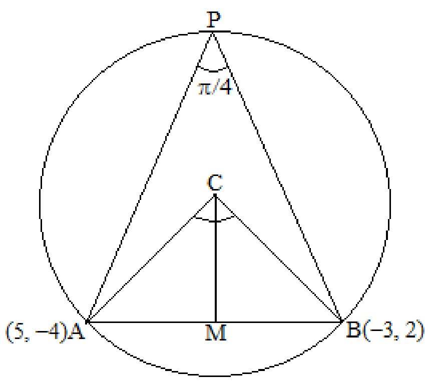
$$\left|\frac{1}{Z_2}+\frac{1}{Z_1}\right|=\left|\frac{1}{Z_2}-\frac{1}{Z_1}\right| \Rightarrow |Z_1+Z_2|=|Z_1-Z_2| \Rightarrow Z_1\overline{Z}_2+Z_2\overline{Z}_1=0.$$

Or $\frac{Z_1}{Z_2}+\frac{\overline{Z}_1}{\overline{Z}_2}=0 \Rightarrow \frac{Z_1}{Z_2}$ is purely imaginary.

Hence $\angle POQ=\frac{\pi}{2}$.

Triangle OPQ is right angled with PQ as hypotenuse.

Circum center will be midpoint of PQ i.e. $\frac{Z_1+Z_2}{2}$.



Q.16 (A), (C), (D)

$AM=10$ & $\angle ACM=\frac{\pi}{4} \Rightarrow CA=\frac{5}{\sin\frac{\pi}{4}}$

or radius $=5\sqrt{2}$.

Also coordinates of M are (1, -1).

Now slope of CM is 4/3 and CM = 5.

Coordinates of C will be $\left(1+5\times\frac{3}{5},-1+5\times\frac{4}{5}\right)$ i.e. $(4,3)$ or

$\left(1-5\times\frac{3}{5},-1-5\times\frac{4}{5}\right)$ i.e. $(-2,-5)$.

Length of arc APB = $5\sqrt{2}\times\frac{3\pi}{2}$ i.e. $\frac{15\pi}{\sqrt{2}}$.

Q.17 (A), (B), (C), (D)

Let $\sqrt{5-12i}=x+iy$, then $x^2-y^2=5$ & $2xy=-12$.

Now $x^2+y^2=\sqrt{\left(x^2-y^2\right)^2+4x^2y^2}=13 \Rightarrow x=\pm 3$ & $y=\pm 2$, but $xy<0$, hence $\sqrt{5-12i}=3-2i$ & $-3+2i$.

Similarly let $\sqrt{-5-12i}=x+iy$, then $x^2-y^2=-5$ & $2xy=-12$.

Now $x^2+y^2=\sqrt{\left(x^2-y^2\right)^2+4x^2y^2}=13 \Rightarrow x=\pm 2$ & $y=\pm 3$, but $xy<0$, hence $\sqrt{5-12i}=2-3i$ & $-2+3i$.

Now possible values of $Z=\sqrt{5-12i}+\sqrt{5-12i}=5(1-i),\ 1+i,\ -5(1+i)$ & $-1+i$.

Possible values of $\operatorname{Arg}(Z)=\frac{\pi}{4},-\frac{\pi}{4},\frac{3\pi}{4},-\frac{3\pi}{4}$.

COMPREHENSION TYPE

Paragraph I

$$|Z_1+Z_2|^2=|Z_1|^2+|Z_2|^2+Z_1\overline{Z}_2+Z_2\overline{Z}_1.$$

Q.1 (b)

$|Z_1+Z_2|^2=|Z_1|^2+|Z_2|^2 \Rightarrow Z_1\overline{Z}_2+Z_2\overline{Z}_1=0$, hence $Z_1\overline{Z}_2$ is purely imaginary.

Q.2 (b)

$Z_1\overline{Z}_2+Z_2\overline{Z}_1=0 \Rightarrow \frac{Z_1}{Z_2}+\frac{\overline{Z}_1}{\overline{Z}_2}=0$, hence $\frac{Z_1}{Z_2}$ is purely imaginary.

Q.3 (c)

As $\frac{Z_1}{Z_2}$ is purely imaginary therefore $i\frac{Z_1}{Z_2}$ is purely real.

Q.4 (c)

As $\frac{Z_1}{Z_2}$ is purely imaginary therefore $\arg\left(\frac{Z_1}{Z_2}\right)$ i.e. $\arg(Z_1)-\arg(Z_2)=\pm\frac{\pi}{2}$.

Paragraph II

$$Z=\frac{1-i\sin\theta}{1+i\sin\theta} \Rightarrow Z=\frac{\cos^2\theta}{1+\sin^2\theta}-\frac{2\sin\theta}{1+\sin^2\theta}i$$

Q.5 (c)

If Z is purely real, then $\frac{2\sin\theta}{1+\sin^2\theta}=0 \Rightarrow \theta=n\pi$.

Q.6 (d)

If Z is purely imaginary, then $\frac{\cos^2\theta}{1+\sin^2\theta}=0 \Rightarrow \theta=(2n+1)\frac{\pi}{2}$.

Q.7 (d)

$|Z|=1 \Rightarrow \cos^4\theta+4\sin^2\theta=(1+\sin^2\theta)^2$ or $\cos^2\theta+\sin^2\theta=1$.

Hence Z is unimodular for all real values of θ.

Q.8 (d)

$\arg(Z)=\frac{\pi}{4} \Rightarrow -\frac{2\sin\theta}{\cos^2\theta}=\tan\frac{\pi}{4}$ or $\sin^2\theta-2\sin\theta-1=0$.

Now $\sin^2\theta - 2\sin\theta - 1 = 0$ gives $\theta = n\pi + (-1)^n \sin^{-1}\left(1-\sqrt{2}\right)$.

Paragraph III

$$\left||Z| - \frac{4}{|Z|}\right| \le \left|Z - \frac{4}{Z}\right| \le |Z| + \frac{4}{|Z|} \Rightarrow \left||Z| - \frac{4}{|Z|}\right| \le 2 \text{ \& } |Z| + \frac{4}{|Z|} \ge 2.$$

Now $|Z| + \frac{4}{|Z|} \ge 2 \Rightarrow |Z|^2 - 2|Z| + 4 \ge 0$, which is true for all $|Z|$.

Similarly $-2 \le |Z| - \frac{4}{|Z|} \le 2 \Rightarrow |Z|^2 + 2|Z| - 4 \ge 0$ & $|Z|^2 - 2|Z| - 4 \le 0$.

Q.9 (a)

$\sqrt{5} - 1 \le |Z| \le \sqrt{5} + 1$.

Hence the difference in least & the greatest values of |Z| is 2.

Q.10 (b)

$$\left|Z - \frac{4}{Z}\right| = 2 \Rightarrow |Z|^2 + \frac{16}{|Z|^2} - 4\frac{Z}{\overline{Z}} - 4\frac{\overline{Z}}{Z} = 4 \;\Rightarrow \left(|Z| - \frac{4}{|Z|}\right)^2 - 4\frac{Z^2}{|Z|^2} - 4\frac{\overline{Z}^2}{|Z|^2} + 4 = 0$$

But for greatest & least $|Z|$, $\left(|Z| - \frac{4}{|Z|}\right)^2 = \left|Z - \frac{4}{Z}\right|^2$. Hence $\frac{Z^2}{|Z|^2} + \frac{\overline{Z}^2}{|Z|^2} = 2$ i.e. $\frac{2\,\mathrm{Re}\left(Z^2\right)}{|Z|^2} = 2$.

or $\mathrm{Re}\left(Z^2\right) = |Z|^2 \Rightarrow \cos 2\theta = 1$ or $\theta = \pi \,\&\, 0$.

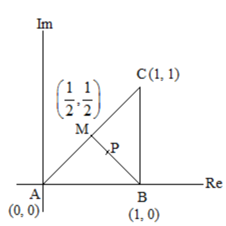
Now $\arg\left(\frac{Z_1}{Z_2}\right) = \arg(Z_1) - \arg(Z_2) = \pi$.

Q.11 (b)

$|Z - Z_1| = |Z - Z_2| \Rightarrow \left|Z - \left(\sqrt{5}+1\right)\right| = \left|Z - \left(\sqrt{5}-1\right)\right|$.

Hence locus of Z will be the line passing through $\left(\sqrt{5}, 0\right)$ and parallel to the Imaginary axis.

Paragraph IV



Let Z = x + iy so that P(Z) be (x, y), then $\frac{y}{x-1} = \frac{0 - \frac{1}{2}}{1 - \frac{1}{2}}$ or $x + y = 1$.

Now $\mu = Z^2 + 1 = x^2 - y^2 + 2ixy$.

Q.12 (a)

$\mu = h + ki$, then $Z^2 + 1 = h + ki \Rightarrow x^2 - y^2 = h - 1, 2xy = k$ & $x + y = 1 \Rightarrow x^2 + y^2 = 1 - k$.

Now $2x^2 = h - k \,\&\, 2y^2 = 2 - h - k$

$\therefore 4x^2y^2 = k^2 \Rightarrow (h-k)(2-h-k) = k^2$ **or** $(h-1)^2 = -2\left(k - \frac{1}{2}\right)$

Hence locus of (μ) is the parabola $(x-1)^2 = -2\left(y - \frac{1}{2}\right)$.

Q.13 (c)

Locus of m is $(x-1)^2 = -2\left(y - \frac{1}{2}\right)$, whose axis is x = 1 i.e. $Z + \overline{Z} = 2$.

Q.14 (b)

Locus of m is $(x-1)^2 = -2\left(y - \frac{1}{2}\right)$, whose directrics is y = 1 i.e. $Z - \overline{Z} = 2i$.

ASSERTION REASONING TYPE

Q.1 (A)

Greatest possible value of principal argument of any complex number is π, hence
$\arg(Z_1Z_2) = 2\pi \Rightarrow \text{Arg}(Z_1) = \text{Arg}(Z_2) = \pi.$
Hence both the statements are true & statement 2 is the correct explanation of statement I.

Q.2 (C)

Standard concept : $1 + \omega^n + \omega^{2n} = \begin{cases} 3, & \text{if } n = 3m \\ 0, & \text{otherwise} \end{cases}$, hence statement 2 is false.

Now $P(x) = x^3 + x^2 + x = x(1 + x + x^2) = x(x-\omega)(x-\omega^2)$, hence roots of $P(x)$ are $0, \omega \,\&\, \omega^2$.

Now let $Q(x) = (1+x)^n - 1 - x^n$, then

$Q(\omega) = (1+\omega)^n - 1 - \omega^n$ or $Q(\omega) = (-\omega^2)^n - 1 - \omega^n$.

As n is odd integer & not a multiple of 3, hence $Q(\omega) = -(1 + \omega^n + \omega^{2n}) = 0$.

Similarly $Q(\omega^2) = 0$ & $Q(0) = 0$.

Hence P(x) divides Q(x) and statement 1 is true.

Q.3 (D)

Now $|Z_1 \pm Z_2|^2 = |Z_1|^2 + |Z_2|^2 \pm 2(Z_1\overline{Z}_2 + Z_2\overline{Z}_1) \,\&\, |Z_1 + Z_2| = |Z_1 - Z_2| \Rightarrow Z_1\overline{Z}_2 + Z_2\overline{Z}_1 = 0$

Or $\frac{Z_1}{Z_2} + \left(\frac{\overline{Z}_1}{\overline{Z}_2}\right) = 0$ which implies $\frac{Z_1}{Z_2}$ is purely imaginary and $\arg\left(\frac{Z_1}{Z_2}\right) = \frac{\pi}{2}$.

Hence the triangle AOB is right angled at O.

Now the point $P\left(\frac{Z_1+Z_2}{2}\right)$ is midpoint of AB, hence its circumcenter.

Q.4 (D)

Statement 2 is a standard property of an ellipse.
Equation of an ellipse having $S(Z_1)\&S'(Z_2)$ as foci and major axis = 2a is
$|Z-Z_1|+|Z-Z_2|=2a$.
But for the given equation distance between (1, 0) & (8, 0) is more than 5, but in ellipse distance between foci is less than the major axis.
Statement 1 is false.

Q.5 (A)

If $1,\alpha,\alpha^2,\ldots,\alpha^{n-1}$ represent n roots of unity, then product of these is 1, hence statement 2 is true.

Similarly if $Z=\left(e^{i\alpha}\right)^{3/5}$, then $Z^5=e^{3i\alpha}$. Let Z_k denote roots of Z for $k=0,1,2,3,4$, then

$Z_k=e^{\frac{2k\pi+3\alpha}{5}i}$ or $Z=e^{\frac{2k\pi}{5}i}e^{\frac{3\alpha}{5}i}$, where $e^{\frac{2k\pi}{5}i}$ denotes 5 roots of unity.

Now $\prod_{k=0}^{4}Z_k=\left(\prod_{k=0}^{4}e^{\frac{2k\pi}{5}i}\right)\left(\prod_{k=0}^{4}e^{\frac{3\alpha}{5}i}\right) \Rightarrow \prod_{k=0}^{4}Z_k=e^{3i\alpha}$ i.e. $\cos 3\alpha+i\sin 3\alpha$. .

Hence both the statements are true & statement 2 is the correct explanation of statement

MATRIX MATCH TYPE

Q.1 $\mathbf{(A)\rightarrow(s),(B)\rightarrow(r),(C)\rightarrow(p),(D)\rightarrow(q)}$

(A) $Z^4=1\Rightarrow Z=\cos\frac{k\pi}{2}+i\sin\frac{k\pi}{2}$ for $k=0,1,2,3$.

(B) $Z^4=-1\Rightarrow Z=\cos\frac{(2k+1)\pi}{4}+i\sin\frac{(2k+1)\pi}{4}$ for $k=0,1,2,3$.

(C) $Z^4=i\Rightarrow Z=\cos\frac{(4k+1)\pi}{8}+i\sin\frac{(4k+1)\pi}{8}$ for $k=0,1,2,3$.

(D) $Z^4=-i\Rightarrow Z=\cos\frac{(4k-1)\pi}{8}+i\sin\frac{(4k-1)\pi}{8}$ for $k=0,1,2,3$.

Q.2 $\mathbf{(A)\rightarrow(q),(B)\rightarrow(s),(C)\rightarrow(p),(D)\rightarrow(r)}$

(A) $P(Z)$ lies on perpendicular bisector of line segment joining $A(1,0)\&B(0,1)$.

(B) $\left|Z+\bar{Z}\right|+\left|Z-\bar{Z}\right|=2\Rightarrow\left|x\right|+\left|y\right|=1,$ where $Z=x+iy.$

(C) $\left|Z+\bar{Z}\right|=\left|Z-\bar{Z}\right|\Rightarrow\left|x\right|=\left|y\right|,$ where $Z=x+iy.$

(D) Let $Z=x+iy$ & $\frac{2}{Z}=h+ik,$ then $h=\frac{2x}{x^2+y^2}$ & $k=-\frac{2k}{x^2+y^2}.$

But $\left|Z\right|=1\Rightarrow x^2+y^2=1,$ hence $h^2+k^2=4.$

Q.3 $(A)\to(p),(r);(B)\to(p),(q),(r),(t);(C)\to(p),(r),(s);(D)\to(p),(q),(r),(s),(t)$

Let Z_1, Z_2, Z_3 & Z_4 represent the points A, B, C & $D.$

(p) $Z_1-Z_4=Z_2-Z_3\Rightarrow\frac{Z_1+Z_3}{2}=\frac{Z_2+Z_4}{2}$, hence AC & BD i.e. diagonals bisect each other.

This property is true in case of a parallelogram, a rhombus, a rectangle or a square.

(q) $\left|Z_1-Z_3\right|=\left|Z_2-Z_4\right|\Rightarrow$ AC = BD i.e. Diagonals are of equal length.

This property is true in case of a rectangle or a square.

(r) $\frac{Z_1-Z_2}{Z_3-Z_4}$ is purely real hence $\arg\left(\frac{Z_1-Z_2}{Z_3-Z_4}\right)=0$ or $\pi.$ Hence AB || CD.

This property is true in case of a parallelogram, a rhombus, a rectangle or a square.

(s) $\frac{Z_1-Z_3}{Z_2-Z_4}$ is purely imaginary hence $\arg\left(\frac{Z_1-Z_3}{Z_2-Z_4}\right)=\pm\frac{\pi}{2}.$ Hence $AC\perp BD.$

This property is true in case ofa rhombus or a square.

(t) $\frac{Z_1-Z_2}{Z_3-Z_2}$ is purely imaginary hence $\arg\left(\frac{Z_1-Z_2}{Z_3-Z_2}\right)=\pm\frac{\pi}{2}.$ Hence $AB\perp BC.$

This property is true in case ofa rectangle or a square.

Q.4 $(A)\to(q),(r);(B)\to(p),(s);(C)\to(q),(s);(D)\to(p),(r)$

(a) $Z^2-Z+1=0$

$\Rightarrow Z=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{3}i}{2}$

$\Rightarrow$ Hence $\frac{\pi}{3}$ or $\frac{-\pi}{3}$

(b) $Z^2+Z+1=0$

$\Rightarrow Z=\frac{-1\pm i\sqrt{3}}{2}$

$\Rightarrow$ Hence, $\frac{2\pi}{3}$ or $\frac{-2\pi}{3}$

(c) $Z^2 = \frac{-1 - i\sqrt{3}}{2} = \omega^2$

$\Rightarrow Z = \omega$ or $Z = -\omega$

$\Rightarrow \frac{2\pi}{3}$ or $\frac{-\pi}{3}$

(d) $Z^2 = \frac{-1 + i\sqrt{3}}{2} = \omega = \omega \times \omega^3 = \omega^4$

$\Rightarrow \therefore Z = \pm\omega^2$

$\Rightarrow$ Hence, principle values of arg (Z) are $\frac{-2\pi}{3}$ or $\frac{\pi}{3}$

Complex Numbers

Exercise – 2(C)

Q.1

Let $Z = x + iy$,

then equation of line becomes $x + y = k$ & equation of circle becomes $x^2 + y^2 - 2x - 4y - 13 = 0$.

Now center of the circle is $(1, 2)$ & radius $= 3\sqrt{2}$.

As the line is a secant hence $\left|\frac{1+2-k}{\sqrt{2}}\right| < 3\sqrt{2}$ or $|k-3| < 6$.

Hence $-3 < k < 9$.

Q.2

If Z_1, Z_2 & Z_3 form an equilateral triangle, then $Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$.

Now $Z_1 = a + i, Z_2 = 1 + bi$ & $Z_3 = 0$

$\Rightarrow a^2 - 1 + 2ai + 1 - b^2 + 2bi = a - b + (1 + ab)i$

Or $a^2 - b^2 = a - b$ & $2a + 2b = 1 + ab$.

Hence $a + b = 1$ or $a = b$.

(i) $a + b = 1$ gives $ab = 1$, hence $a^2 - a + 1 = 0$, which doesnt have real roots.

(ii) $a = b$ gives $a^2 - 4a + 1 = 0$, hence $a = b = 2 - \sqrt{3}$ (as a & $b < 1$)

Now $a^2 + b^2 = 2\left(2 - \sqrt{3}\right)^2 = 14 - 8\sqrt{3}$.

As $0 < 14 - 8\sqrt{3} < 1$, hence $\left[a^2 + b^2\right] = 0$.

Q.3

$Z = \frac{\sqrt{3} - i}{2}$ or $Z = i\omega^2 \Rightarrow Z^3 = -i$ & $Z^2 = -\omega$, where ω & ω^2 are complex cube roots of unity.

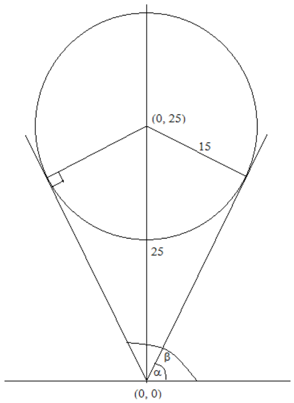
Now $Z^{95} = Z^{3\times31} \times Z^2 = -i\omega$ & $i^{67} = i^{4\times16} \times i^3 = -i$

$\left(Z^{95} + i^{67}\right)^{94} = i^{94}(\omega + 1)^{94} = -\omega^{188}$

$\Rightarrow \left(Z^{95} + i^{67}\right)^{94} = -\omega^2$.

Now $Z^n = i^n\omega^{2n}$, hence least possible value of n is 10.

Q.4



$|Z-25i| \le 15 \Rightarrow P(Z)$ lies on a circle of radius 15 & having center at $(0,25)$.

Let Max.Arg$(Z)=\alpha$ & Min.Arg$(Z)=\beta$, then clearly $\tan\alpha$ & $\tan\beta$ will be slopes of tangents to this circle from O.

As shown in figure $\sin(90-\alpha)=\dfrac{15}{25}$ or $\cos\alpha=\dfrac{3}{5} \Rightarrow \alpha=\cos^{-1}\dfrac{3}{5}$.

Similarly $\sin(\beta-90)=\dfrac{3}{5}$ or $\cos\beta=-\dfrac{3}{5} \Rightarrow \beta=\pi-\cos^{-1}\dfrac{3}{5}$.

Now $|\beta-\alpha| = \pi-2\cos^{-1}\dfrac{3}{5}=2\cos^{-1}\dfrac{4}{5}$.

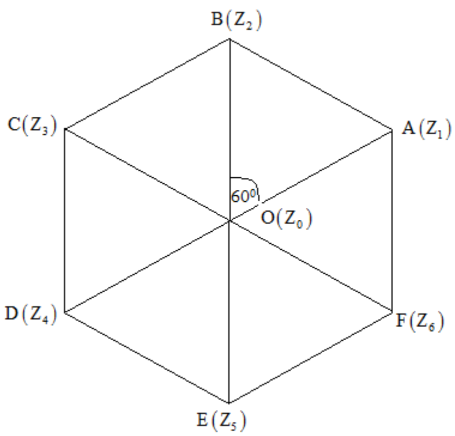
Q.5

Let $Z_i=\cos\theta_i+i\sin\theta_i$ (where $\theta_i=\alpha,\beta,\gamma$), then as given $Z_1+Z_2+Z_3=0$.

$\Rightarrow Z_1^3+Z_2^3+Z_3^3=3Z_1Z_2Z_3$

$\Rightarrow \cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$, hence $\lambda=1$.

Q.6



As Z_0, Z_i & Z_{i+1} form an equilateral triangle hence

$Z_0^2+Z_i^2+Z_{i+1}^2=Z_0Z_i+Z_0Z_{i+1}+Z_iZ_{i+1}$. Hence

$$\sum_{i=1}^{6}\left(Z_0^2+Z_i^2+Z_{i+1}^2\right)=\sum_{i=1}^{6}\left(Z_0Z_i+Z_0Z_{i+1}+Z_iZ_{i+1}\right),$$

where $Z_7=Z_1$.

$$\Rightarrow 6Z_0^2+2\sum_{i=1}^{6}Z_i^2=2Z_0\sum_{i=1}^{6}Z_i$$

Also $Z_0=\dfrac{\sum_{i=1}^{6}Z_i}{6} \Rightarrow \sum_{i=1}^{6}Z_i^2=3Z_0^2$.

Q.7

$\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11} = -i\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right)$. But $\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}$, for $q = 0, 1, 2, ..., 10$

gives 11th cube roots of unity, hence $\sum_{q=0}^{10}\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right) = 0.$

Now $\sum_{i=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right) = (-i)\sum_{i=1}^{10}\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right)$

Or $\sum_{i=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right) = i - i\sum_{i=0}^{10}\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right)^p = i.$

Now $\sum_{p=1}^{32}(3p+2)\left(\sum_{i=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p = \sum_{p=1}^{32}(3p+2)i^p$

Let $S = 5i + 8i^2 + 11i^3 + ... + 98i^{32}$, then

$iS = \quad 5i^2 + 8i^3 + ... + 95i^{32} + 98i^{33}$

$(1-i)S = 5i + 3i^2 + 3i^3 + ...3i^{32} - 98i^{33}$

Or $S = \frac{-96i}{1-i} + \frac{3i(1-i^{32})}{(1-i)^2} \Rightarrow S = \frac{-96i(1+i)}{2} = 48 - 48i$, hence $a + b = 0$.

Q.8

$Z^6 + Z^5 + Z^4 + Z^3 + Z^2 + Z + 1 = 0 \Rightarrow Z^7 - 1 = 0, Z \neq 1.$

Hence $Z_i, i = 1, 2, ..., 6$ are 7th roots of unity except 1 itself.

Now $\sum_{i=0}^{6} Z_i^{\,r} = \begin{cases} 7 & \text{if } r = 7k \\ 0 & \text{otherwise} \end{cases}$, hence $\sum_{i=0}^{6} Z_i^{\,5} = 0$ & $\sum_{i=0}^{6} Z_i^{\,14} = 7.$

$\Rightarrow \sum_{i=1}^{6} Z_i^{\,5} = -1$ & $\sum_{i=1}^{6} Z_i^{\,14} = 6.$

Also from the given equation $\prod_{i=1}^{6} Z_i = 1.$

Now $\sum_{i=1}^{6} Z_i^{\,5} + \sum_{i=1}^{6} Z_i^{\,14} - \prod_{i=1}^{6} Z_i = -1 + 6 - 1 = 4.$

Q.9

$\left||z| - \frac{4}{|z|}\right| \leq \left|z - \frac{4}{z}\right| \Rightarrow \left||z| - \frac{4}{|z|}\right| \leq 3$

$\Rightarrow -3 \leq |z| - \frac{4}{|z|} \leq 3$

$\Rightarrow |z|^2 - 3|z| - 4 \le 0$ & $|z|^2 + 3|z| - 4 \ge 0$

$\Rightarrow 1 \le |z| \le 4$

Q.10

$z^3 = 343 \Rightarrow (z-7)(z^2+7z+49) = 0$

Hence $z^2 + 7z + 49 \equiv z^2 + az + b$

$\Rightarrow a = 7, b = 49$

$\Rightarrow \dfrac{7a+b}{14} = 7$

Q.11

$|z-4| = |z-8| \Rightarrow (z-4)(\bar{z}-4) = (z-8)(\bar{z}-8)$

$\Rightarrow z + \bar{z} = 12 \Rightarrow \text{Re}(z) = 6$

Let $z = 6 + ai$

Now $3|z-12| = 5|z-8i| \Rightarrow 3|ai-6| = 5|6+(a-8)i|$

$\Rightarrow 9(a^2+36) = 25(a^2-16a+100)$

$\Rightarrow a^2 - 25a + 136 = 0$

$\Rightarrow a = 8, 17$

Hence $\text{Im}(z) = 8$

Q.12

Let $a = a_1 + a_2 i$ & $b = b_1 + b_2 i$

$f(z) = (4+i)z^2 + (a_1 + a_2 i)z + b_1 + b_2 i$

$f(1) = a_1 + a_2 i + b_1 + b_2 i + (4+i)$

$f(i) = a_1 i - a_2 + b_1 + b_2 i - (4+i)$

$f(1) = \overline{f(1)} \Rightarrow a_1 + a_2 i + b_1 + b_2 i + (4+i) = a_1 - a_2 i + b_1 - b_2 i + (4-i)$

$\Rightarrow a_2 + b_2 = -1$ ...(i)

$f(i) = \overline{f(i)} \Rightarrow a_1 i - a_2 + b_1 + b_2 i - (4+i) = -a_1 i - a_2 + b_1 - b_2 i - (4-i)$

$\Rightarrow a_1 + b_2 = 1$ ...(ii)

$\Rightarrow a_1^2 + a_2^2 = 2 + 2b_2^2$

$\Rightarrow |a| + |b| = \sqrt{2(1+b_2^2)} + \sqrt{b_1^2 + b_2^2}$

$\Rightarrow |a| + |b| \ge \sqrt{2}$

Q.13

$z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$ ...(i)

$\Rightarrow \bar{z}^4 + a_1 \bar{z}^3 + a_2 \bar{z}^2 + a_3 \bar{z} + a_4 = 0$

If z is purely imaginary, then $\bar{z} = -z$

$\Rightarrow z^4 - a_1 z^3 + a_2 z^2 - a_3 z + a_4 = 0$ ...(ii)

From (i) & (ii)

$z^4 + a_2 z^2 + a_4 = 0$ & $a_1 z^2 + a_3 = 0$

$$\Rightarrow \left(\frac{a_3}{a_1}\right)^2 - a_2\left(\frac{a_3}{a_1}\right) + a_4 = 0$$

$$\Rightarrow \frac{a_3}{a_1 a_2} + \frac{a_1 a_4}{a_2 a_3} = 1$$

Q.14

$z^3 + \left(\overline{w}\right)^7 = 0$ & $z^5 w^{11} = 1 \Rightarrow |z| = |w| = 1$

Now $z^3 + \left(\overline{w}\right)^7 = 0 \Rightarrow z^3 w^7 = -1$

Further $z^3 w^7 = -1$ & $z^5 w^{11} = 1$

$\Rightarrow z = w = \mp i$

Hence $(z, w) \equiv (i, i)$ or $(-i, -i)$

Q.15

$$\frac{z_1^2}{z_2 z_3} + \frac{z_2^2}{z_1 z_3} + \frac{z_3^2}{z_1 z_2} + 1 = 0 \Rightarrow z_1^3 + z_2^3 + z_3^3 - 3z_1 z_2 z_3 = -4z_1 z_2 z_3$$

$$\Rightarrow (z_1 + z_2 + z_3)\left(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1\right) = -4z_1 z_2 z_3$$

$$\Rightarrow (z_1 + z_2 + z_3)^3 - 3(z_1 + z_2 + z_3)(z_1 z_2 + z_2 z_3 + z_3 z_1) + 4z_1 z_2 z_3 = 0$$

Let $z = z_1 + z_2 + z_3$, then

$$z^3 - 3z(z_1 z_2 + z_2 z_3 + z_3 z_1) + 4z_1 z_2 z_3 = 0$$

$$\Rightarrow z^3 = z_1 z_2 z_3\left\{3z\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) - 4\right\}$$

$$\Rightarrow z^3 = z_1 z_2 z_3\left\{3z\left(\bar{z}_1 + \bar{z}_2 + \bar{z}_3\right) - 4\right\}$$

$$\Rightarrow |z|^3 = 3|z|^2 - 4$$

$$\Rightarrow |z| = 1, 2$$

Complex Numbers

Exercise – 3

Q.1

$x+1=\sqrt{2}i \Rightarrow (x+1)^4 = 4$ Or $x^4+4x^3+6x^2+4x+9=12.$

Q.2

$a=\dfrac{1+i}{\sqrt{2}} \Rightarrow a^2 = i$, hence $a^6+a^4+a^2+1=i^6+i^4+i^2+1=0.$

Q.3

Let $Z_1 = a+i \,\&\, Z_2 = b+i$, then $\left(1+a^2\right)\left(1+b^2\right)=|Z_1|^2|Z_2|^2$

$\Rightarrow \left(1+a^2\right)\left(1+b^2\right)=\left(Z_1\overline{Z}_1\right)\left(Z_2\overline{Z}_2\right)=\left(Z_1\overline{Z}_2\right)\left(Z_2\overline{Z}_1\right)$

$\Rightarrow \left(1+a^2\right)\left(1+b^2\right)=\left(ab+1+(b-a)i\right)\left(ab+1-(b-a)i\right)$

Or $\left(1+a^2\right)\left(1+b^2\right)=(ab+1)^2+(b-a)^2.$

Q.4

Let $\sqrt{7-24i}=a+bi$, then $7-24i=a^2-b^2+2abi$

$\Rightarrow a^2-b^2=7 \;\&\; 2ab=-24.$

Now $\left(a^2+b^2\right)^2=\left(a^2-b^2\right)^2+4a^2b^2 \Rightarrow a^2+b^2=25.$

From $a^2+b^2=25 \;\&\; a^2-b^2=7$, we get $a^2=16 \;\&\; b^2=9.$

As $ab<0$, hence $a=4 \,\&\, b=-3$ or $a=-4 \,\&\, b=3.$

Required square roots are $4-3i \;\&\; -4+3i.$

Q.5

$Z_1=3i \Rightarrow \arg(Z_1)=\dfrac{\pi}{2} \;\&\; Z_2=-1-i \Rightarrow \arg(Z_2)=-\dfrac{3\pi}{4}.$

Now $\arg\left(\dfrac{Z_1}{Z_2}\right)=\arg(Z_1)-\arg(Z_2)=\dfrac{5\pi}{4}$

Now $\dfrac{5\pi}{4}$ is an angle in third quadrant, hence principal argument will be $-\dfrac{3\pi}{4}.$

Q.6

Let $Z = x + yi$, then $Z^2 + |Z| = 0 \Rightarrow x^2 - y^2 + \sqrt{x^2 + y^2} + 2ixy = 0$, hence $xy = 0$ & $x^2 - y^2 + \sqrt{x^2 + y^2} = 0$.

$y = 0 \Rightarrow |x| = -x^2$, which is not possible as $x \in R$ and $x = 0$ has already been considered.

$x = 0 \Rightarrow |y| = y^2 \Rightarrow y = -1, 1$ or 0.

Required solution set is $Z = 0, -i, i$.

Q.7

Let $Z = x + yi$, then $Z^2 + \overline{Z} = 0 \Rightarrow x^2 - y^2 + x + (2xy - y)i = 0$

Hence $2xy - y = 0$ & $x^2 - y^2 + x = 0$.

$y = 0 \Rightarrow x^2 + x = 0$, which gives $x = 0, -1$

$x = \frac{1}{2} \Rightarrow y^2 = \frac{3}{4}$, which gives $y = -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}$.

Required solution set is $Z = 0, -1, \frac{1 - \sqrt{3}i}{2}, \frac{1 + \sqrt{3}i}{2}$.

Q.8

Let $Z = x + yi$, then $Z^2 = \left(i\overline{Z}\right)^2 \Rightarrow x^2 - y^2 + 2xyi = y^2 - x^2 + 2xyi$

Hence $x^2 - y^2 = 0 \Rightarrow x = y$ or $x = -y$.

Hence $Z = x + xi$ or $x - xi$.

Q.9

Let $Z = x + yi$, then $|Z + 1| = Z + 2 + 2i \Rightarrow \sqrt{(x+1)^2 + y^2} = (x + 2) + (y + 2)i$

Hence $\sqrt{(x+1)^2 + y^2} = (x + 2)$ & $y + 2 = 0$.

$y = -2$ gives $(x + 2)^2 - (x + 1)^2 = 4 \Rightarrow x = \frac{1}{2}$.

Required solution is $Z = \frac{1}{2} - 2i$.

Q.10

Let $Z = x + yi$, then $i\overline{Z} = Z^2 \Rightarrow y + xi = x^2 - y^2 + 2xyi$, hence $x^2 - y^2 = y$ & $2xy = x$.

$2xy = x \Rightarrow x = 0$ or $y = \frac{1}{2}$. Now $y = \frac{1}{2}$ gives $x = \pm\frac{\sqrt{3}}{2}$ & $x = 0$ gives $y^2 = -y$ i.e. $y = -1$.

Required solutions are $Z = -i, \frac{1 \pm \sqrt{3}}{2}$.

Q.11

Let $Z = x + yi$, then $2|Z|^2 + Z^2 - 5 + \sqrt{3}i \Rightarrow 3x^2 + y^2 + 2xyi = 5 - \sqrt{3}i$

Hence $3x^2 + y^2 = 5$ & $2xy = -\sqrt{3}$.

Now $\left(3x^2 - y^2\right)^2 = \left(3x^2 + y^2\right)^2 - 12x^2y^2 \Rightarrow 3x^2 - y^2 = 4$ or -4.

$3x^2 + y^2 = 5$ & $3x^2 - y^2 = 4$ gives $x^2 = \frac{3}{2}$ & $y^2 = \frac{1}{2} \Rightarrow Z = \pm\frac{\sqrt{3} - i}{2}$ (as $xy < 0$)

$3x^2 + y^2 = 5$ & $3x^2 - y^2 = -4$ gives $x^2 = \frac{1}{6}$ & $y^2 = \frac{9}{2} \Rightarrow Z = \pm\left(\frac{1 - 3\sqrt{3}}{\sqrt{6}}\right)$ (as $xy < 0$).

Q.12

(i) Let $A(-6,0)$ & $B(0,-4)$ be two points on argand plane & $P(Z)$ be a moving a point.

Now $|Z+6| = PA$ & $|Z+4i| = PB$.

$\left|\frac{Z+6}{Z+4i}\right| = \frac{5}{3} \Rightarrow \frac{PA}{PB} = \frac{5}{3}$, hence P will trace a circle.

(ii) Let $A(-2,0)$ & $B(-4,0)$ be two points on argand plane & $P(Z)$ be a moving a point.

Now $|Z+2| = PA$ & $|Z+4| = PB$.

$\left|\frac{Z+2}{Z+4}\right| = 1 \Rightarrow PA = PB$, hence P will trace a straight line.

Q.13

Let $Z = x + yi$, then $Z + c|Z+1| + i = 0 \Rightarrow x + iy + c\sqrt{(x+1)^2 + y^2} + i = 0$

Hence $c\sqrt{(x+1)^2 + y^2} = -x$ & $y = -1$. Now $y = -1$ gives $c^2(x+1)^2 = x^2 - c^2$

or $\left(c^2 - 1\right)x^2 + 2c^2x + 2c^2 = 0 \Rightarrow x = \frac{-c^2 \pm \sqrt{2c^2 - c^4}}{c^2 - 1}$, where $1 < c \le \sqrt{2}$. For $c = 1$, we get $x = -1$.

Hence $Z = \frac{-c^2 \pm \sqrt{2c^2 - c^4}}{c^2 - 1} - i$ for $1 < c \le \sqrt{2}$ & $Z = -1 - i$ for $c = 1$.

Q.14

Let $Z = x + yi$, then $Z^2 = \overline{Z} \Rightarrow x^2 - y^2 + 2xyi = x - iy$. Hence $x^2 - y^2 = x$ & $2xy = -y$.

Now $y = 0$ gives $x = 0$ & 1 & $x = -\frac{1}{2}$ gives $y = \pm\frac{\sqrt{3}}{2}$

Hence $Z = 0, 1, \frac{-1 \pm \sqrt{3}i}{2}$.

Q.15

$$\arg\left(\frac{3Z-6-3i}{2Z-8-6i}\right)=\frac{\pi}{4}\Rightarrow \arg\left(\frac{Z-2-i}{Z-4-3i}\right)=\frac{\pi}{4}$$

Now let $A(2,1)$ & $B(4,3)$ be two fixed point & $P(Z)$, then

$\arg\left(\frac{Z-2-i}{Z-4-3i}\right)=$ angle between PA & PB.

As AB subtends a constant acute angle at P, hence locus of P will be major arc of a circle passing through A & B.

Q.16

$$|Z+6|=|2Z+3|\Rightarrow (x+6)^2+y^2=(2x+3)^2+4y^2 \text{ or } x^2+y^2=9.$$

Q.17

Let $Z=\cos\alpha+i\sin\alpha$, then $Z^2+\overline{Z}=\cos 2\alpha+\cos\alpha+i(\sin 2\alpha-\sin\alpha)$

Or $Z^2+\overline{Z}=2\cos\frac{3\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)$

Now $\arg\left(Z^2+\overline{Z}\right)=\frac{\alpha}{2}=\frac{1}{2}\arg(Z)$.

Q.18

(i) $\prod_{r=1}^{n}(a_r+ib_r)=x+iy \Rightarrow \prod_{r=1}^{n}|a_r+ib_r|^2=x^2+y^2$ or $\prod_{r=1}^{n}\left({a_r}^2+{b_r}^2\right)=x^2+y^2$.

(ii) $\arg\left\{\prod_{r=1}^{n}(a_r+ib_r)\right\}=\sum_{r=1}^{n}\arg(a_r+ib_r)\Rightarrow \sum_{r=1}^{n}\tan^{-1}\left(\frac{b_r}{a_r}\right)=\tan^{-1}\left(\frac{y}{x}\right)$.

Q.19

$|Z_r|=1\Rightarrow Z_r=\frac{1}{\overline{Z_r}}$. Also $|Z|=|\overline{Z}|$.

Now $|Z_1+Z_2+\ldots+Z_n|=\left|\frac{1}{\overline{Z_1}}+\frac{1}{\overline{Z_2}}+\ldots+\frac{1}{\overline{Z_n}}\right|=\left|\frac{1}{Z_1}+\frac{1}{Z_2}+\ldots+\frac{1}{Z_n}\right|$.

Q.20

$$iZ^3+Z^2-Z+i=0\Rightarrow (iZ+1)\left(Z^2+i\right)=0$$

Hence $Z=i,\pm\sqrt{-i}\Rightarrow |Z|=1$.

Q.21

Let affixes of the points $P, Q \& O$ on argand plane be $Z_1, Z_2 \& 0$.

Now $|Z_1 + Z_2|^2 = |Z_1|^2 + |Z_2|^2 + 2\cos\theta$, where $\theta = \angle POQ = \arg\left(\frac{OP}{OQ}\right) = \arg\left(\frac{Z_1}{Z_2}\right)$.

Clearly if $|Z_1 + Z_2|^2 = |Z_1|^2 + |Z_2|^2$, then $\theta = \frac{\pi}{2}$.

Q.22

$|Z_1 + Z_2|^2 = |Z_1 + Z_2||\overline{Z}_1 + \overline{Z}_2| = |Z_1|^2 + |Z_2|^2 + Z_1\overline{Z}_2 + \overline{Z}_1 Z_2$ &

$|Z_1 - Z_2|^2 = |Z_1 - Z_2||\overline{Z}_1 - \overline{Z}_2| = |Z_1|^2 + |Z_2|^2 - Z_1\overline{Z}_2 - \overline{Z}_1 Z_2$

$\Rightarrow |Z_1 + Z_2|^2 + |Z_1 - Z_2|^2 = 2|Z_1|^2 + 2|Z_2|^2$.

Q.23

By $|Z_1 - Z_2|^2 = |Z_1|^2 + |Z_2|^2 - Z_1\overline{Z}_2 - \overline{Z}_1 Z_2$,

$|1 - \overline{Z}_1 Z_2|^2 = 1 + |\overline{Z}_1 Z_2|^2 - Z_1\overline{Z}_2 - \overline{Z}_1 Z_2$ or $|1 - \overline{Z}_1 Z_2|^2 = 1 + |Z_1|^2|Z_2|^2 - Z_1\overline{Z}_2 - \overline{Z}_1 Z_2$

Now $|1 - \overline{Z}_1 Z_2|^2 - |Z_1 - Z_2|^2 = 1 + |Z_1|^2|Z_2|^2 - |Z_1|^2 - |Z_2|^2$

$\Rightarrow |1 - \overline{Z}_1 Z_2|^2 - |Z_1 - Z_2|^2 = \left(1 - |Z_1|^2\right)\left(1 - |Z_2|^2\right)$.

Q.24

$|iZ + 3 - 5i| = |Z - 5 - 3i|$, Now using $|Z_1 + Z_2| \le |Z_1| + |Z_2|$ we get

$|Z - 5 - 3i| \le |Z - 1| + |4 + 3i| \Rightarrow |Z - 5 - 3i| < 8$.

Q.25

$|(1+i)Z^3 + iZ| \le |Z^3| + |iZ^3| + |iZ| \Rightarrow |(1+i)Z^3 + iZ| \le 2|Z|^3 + |Z|$

Now $|Z| < \frac{1}{2} \Rightarrow |(1+i)Z^3 + iZ| < \frac{3}{4}$.

Q.26

$Z^2 + \alpha Z + \beta = 0 \ldots (i) \Rightarrow \overline{Z}^2 + \overline{\alpha}\overline{Z} + \overline{\beta} = 0$.

As Z is real, $Z = \overline{Z}$, hence $Z^2 + \overline{\alpha} Z + \overline{\beta} = 0 \ldots (ii)$

Applying the condition of common root on $(i)\&(ii)$ gives

$$(\bar{\alpha}-\alpha)(\alpha\bar{\beta}-\bar{\alpha}\beta)=(\beta-\bar{\beta})^2 \text{ or } \frac{(\bar{\beta}-\beta)^2}{(\alpha-\bar{\alpha})^2}+\frac{(\bar{\beta}-\beta)}{(\alpha-\bar{\alpha})}+\beta=0.$$

Q.27

$$Z^2+aZ+b=0\Rightarrow\left(Z+\frac{a}{2}\right)^2=\frac{a^2}{4}-b.$$

Clearly if $\frac{a^2}{4}-b\ge 0$, then Z must be real, hence for nonreal complex roots $a^2<4b$.

Q.28

By properties of modulus, $\left||Z|-\frac{1}{|Z|}\right|\le\left|Z+\frac{1}{Z}\right|\le|Z|+\frac{1}{|Z|}\ \Rightarrow\ |Z|+\frac{1}{|Z|}\ge 2\ \&\ \left||Z|-\frac{1}{|Z|}\right|\le 2.$

Now first inequality is by itself true as $|Z|+\frac{1}{|Z|}\ge 2\sqrt{|Z|\times\frac{1}{|Z|}}$ by A.M. $\ge$ G.M.

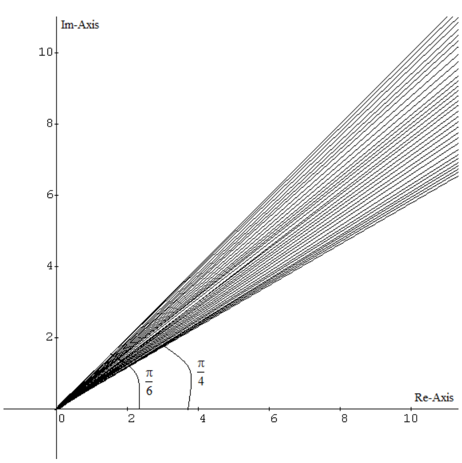
$$\left||Z|-\frac{1}{|Z|}\right|\le 2\ \Rightarrow -2\le|Z|-\frac{1}{|Z|}\le 2.$$

(i) $|Z|-\frac{1}{|Z|}\le 2\Rightarrow|Z|^2-2|Z|-1\le 0$ or $|Z|\le 1+\sqrt{2}.$

(ii) $|Z|-\frac{1}{|Z|}\ge -2\Rightarrow|Z|^2+2|Z|-1\ge 0$ or $|Z|\ge\sqrt{2}-1.$

Hence $\sqrt{2}-1\le|Z|\le 1+\sqrt{2}.$

Q.29



Q.30

$x^2 - 2x + 1 = -3 \Rightarrow (x-1)^2 = \sqrt{3}\,i$ or $x = 1 \pm \sqrt{3}\,i$.

Hence $x = 2e^{\pm\frac{i\pi}{3}}$, which implies $Z_1{}^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$ & $Z_2{}^n = 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$

Now $Z_1{}^n + Z_2{}^n = 2^n\left(2\cos\frac{n\pi}{3}\right)$.

Q.31

Let A & B be the points $(-1,0)$ & $(1,0)$ & $P(Z)$ be a moving point, then

$|Z+1|^2 + |Z+1|^2 = 2^2 \Rightarrow PA^2 + PB^2 = AB^2$.

Hence P lies on the circle having AB as diameter.

Q.32

Let $Z_k = \cos\theta_k + i\sin\theta_k$, then $\sum_{k=1}^{3}\cos\theta_k = 0 = \sum_{k=1}^{3}\sin\theta_k \Rightarrow Z_1 + Z_2 + Z_3 = 0$.

(i) $Z_1 + Z_2 + Z_3 = 0 \Rightarrow Z_1{}^2 + Z_2{}^2 + Z_3{}^2 = (Z_1 + Z_2 + Z_3)^2 - 2(Z_1Z_2 + Z_2Z_3 + Z_3Z_1)$.

Now $Z_1Z_2 + Z_2Z_3 + Z_3Z_1 = Z_1Z_2Z_3\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)$

$\therefore\ Z_1Z_2 + Z_2Z_3 + Z_3Z_1 = Z_1Z_2Z_3\left(\bar{Z}_1 + \bar{Z}_2 + \bar{Z}_3\right) = 0$ as $|Z_k| = 1$ and hence $\bar{Z}_k = \frac{1}{Z_k}$.

$\Rightarrow Z_1{}^2 + Z_2{}^2 + Z_3{}^2 = 0$ or $\sum_{k=1}^{3}(\cos 2\theta_k + i\sin 2\theta_k) = 0$.

Hence $\cos 2\theta_1 + \cos 2\theta_2 + \cos 2\theta_3 = 0 = \sin 2\theta_1 + \sin 2\theta_2 + \sin 2\theta_3$

(ii) $Z_1 + Z_2 + Z_3 = 0 \Rightarrow Z_1{}^3 + Z_2{}^3 + Z_3{}^3 = 3Z_1Z_2Z_3$.

$\therefore \sum_{k=1}^{3}(\cos 3\theta_k + i\sin 3\theta_k) = 3\cos(\theta_1 + \theta_2 + \theta_3) + 3i\sin(\theta_1 + \theta_2 + \theta_3)$

Hence $\cos 3\theta_1 + \cos 3\theta_2 + \cos 3\theta_3 = 3\cos(\theta_1 + \theta_2 + \theta_3)$

Q.33

Let $\frac{1}{Z+a} + \frac{1}{Z+b} + \frac{1}{Z+c} + \frac{1}{Z+d} = \frac{2}{Z}$, then $\frac{1}{\bar{Z}+a} + \frac{1}{\bar{Z}+b} + \frac{1}{\bar{Z}+c} + \frac{1}{\bar{Z}+d} = \frac{2}{\bar{Z}}$

Clearly if ω is a root, then $\bar{\omega}$ i.e. ω^2 must also be a root as $a, b, c, d \in R$.

$\therefore \frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} + \frac{1}{d+\omega^2} = \frac{2}{\omega^2}$ ...(i)

Now $(Z+b)(Z+c)(Z+d)Z+(Z+a)(Z+c)(Z+d)Z+(Z+a)(Z+b)(Z+d)Z$
$+(Z+a)(Z+b)(Z+c)Z=2(Z+a)(Z+b)(Z+c)(Z+d)$
or $2Z^4+(a+b+c+d)Z^3-(abc+bcd+cda+dab)Z-2abcd=0$

Let the roots be $\omega,\omega^2,\alpha\,\&\,\beta$, then
$\omega.\omega^2+\omega^2.\alpha+\alpha.\beta+\omega.\alpha+\omega.\beta+\omega^2.\beta=0\Rightarrow 1+\alpha.\beta-\alpha-\beta=0.$
Hence 1 must also be a root $(\text{let } \beta=1)$.
$\therefore \dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1}+\dfrac{1}{d+1}=2$ ...(iv)
Further $\omega.\omega^2.\alpha.\beta=-abcd\Rightarrow\alpha=-abcd$
& $\omega+\omega^2+\alpha+\beta=-\dfrac{a+b+c+d}{2}\Rightarrow\alpha=-\dfrac{a+b+c+d}{2}$
$\therefore a+b+c+d=2abcd$ ...(iii)
Lastly $\omega.\omega^2.\alpha+\omega^2.\alpha.\beta+\alpha.\beta.\omega+\beta.\omega.\omega^2=-\dfrac{abc+bcd+cda+dab}{2}\Rightarrow abc+bcd+cda+dab=2$ ...(ii)

Q.34

(i)
$(1+x)^n=a_0+a_1x+a_2x^2+a_3x^3+\dots+a_nx^n$, then by substituting i & $-i$,
$(1+i)^n=a_0+a_1i-a_2-a_3i+a_4+a_5i-a_6-a_7i+\dots$
$(1-i)^n=a_0-a_1i-a_2+a_3i+a_4-a_5i-a_6+a_7i+\dots$
Adding the two gives $2(a_0-a_2+a_4-a_6+\dots)=(1+i)^n+(1-i)^n$.
Now $1+i=\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)\Rightarrow(1+i)^n+(1-i)^n=2\left(\sqrt{2}\right)^n\cos\dfrac{n\pi}{4}$.
Hence $a_0-a_2+a_4-a_6+\dots=2^{n/2}\cos\dfrac{n\pi}{4}$.

(ii)
$(1+x)^n=a_0+a_1x+a_2x^2+a_3x^3+\dots+a_nx^n$, then by substituting i & $-i$,
$(1+i)^n=a_0+a_1i-a_2-a_3i+a_4+a_5i-a_6-a_7i+\dots$
$(1-i)^n=a_0-a_1i-a_2+a_3i+a_4-a_5i-a_6+a_7i+\dots$

Subtracting the two gives $2i(a_1 - a_3 + a_5 - a_7 + \ldots) = (1+i)^n + (1-i)^n$.

Now $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow (1+i)^n - (1-i)^n = 2\left(\sqrt{2}\right)^n i\sin\frac{n\pi}{4}$.

Hence $a_1 - a_3 + a_5 - a_7 + \ldots = 2^{n/2}\sin\frac{n\pi}{4}$.

(iii)

Now $(a_0 - a_2 + a_4 - a_6 + \ldots)^2 + (a_1 - a_3 + a_5 - a_7 + \ldots)^2 = 2^n$.

(iv)

$(1+x)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$, then by substituting 1, ω & ω^2,

$2^n = a_0 + a_1 + a_2 + a_3 + \ldots$

$(1+\omega)^n = a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + a_5\omega^2 + a_6 + \ldots$

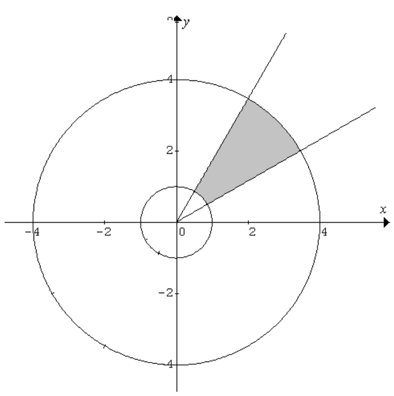
$\left(1+\omega^2\right)^n = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + a_5\omega + a_6 + \ldots$

Adding the three relations gives $3(a_0 + a_3 + a_6 + \ldots) = 2^n + \left(-\omega^2\right)^n + (-\omega)^n$

Now $-\omega = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} \Rightarrow \left(-\omega^2\right)^n + (-\omega)^n = 2\cos\frac{n\pi}{3}$.

Hence $a_0 + a_3 + a_6 + \ldots = \dfrac{2^n + 2\cos\frac{n\pi}{3}}{3}$.

Q.35



Required area is area of shaded sector as shown in the figure.

Area of sector of angle $\frac{\pi}{6}$ of circle of radius 4 will be $\frac{4\pi}{3}$.

Area of sector of angle $\frac{\pi}{6}$ of circle of radius 1 will be $\frac{\pi}{12}$.

Hence required area $= \frac{4\pi}{3} - \frac{\pi}{12}$ i.e. $\frac{5\pi}{4}$.

Q.36

$\left|Z_1 - Z_2\right|^2 = \left|Z_1\right|^2 + \left|Z_2\right|^2 - \left(Z_1 \overline{Z}_2 + Z_2 \overline{Z}_1\right)$

or $\left|Z_1 - Z_2\right|^2 = \left|Z_1\right|^2 + \left|Z_2\right|^2 - 2\left|Z_1\right|\left|Z_2\right|\cos\theta$, where $\theta = \arg\left(Z_1\right) - \arg\left(Z_2\right)$

$\Rightarrow \left|Z_1 - Z_2\right|^2 = \left|Z_1\right|^2 + \left|Z_2\right|^2 - 2\left|Z_1\right|\left|Z_2\right| + 2\left|Z_1\right|\left|Z_2\right|\left(1 - \cos\theta\right)$

$\Rightarrow \left|Z_1 - Z_2\right|^2 = \left(\left|Z_1\right| - \left|Z_2\right|\right)^2 + 4\left|Z_1\right|\left|Z_2\right|\sin^2\dfrac{\theta}{2}.$

Now $\left|\sin\theta\right| \le \left|\theta\right| \Rightarrow \sin^2\dfrac{\theta}{2} \le \dfrac{\theta^2}{4}$, hence

$\left|Z_1 - Z_2\right|^2 \le \left(\left|Z_1\right| - \left|Z_2\right|\right)^2 + \theta^2$ or $\left|Z_1 - Z_2\right|^2 \le \left(\left|Z_1\right| - \left|Z_2\right|\right)^2 + \left(\arg\left(Z_1\right) - \arg\left(Z_2\right)\right)^2.$

Q.37

$Z^7 + 4Z^3 + 11 = 0 \Rightarrow \left|Z^7 + 4Z^3\right| = 11.$

But $\left|\left|Z\right|^7 - 4\left|Z\right|^3\right| \le \left|Z^7 + 4Z^3\right| \le \left|Z\right|^7 + 4\left|Z\right|^3$ or $\left|Z\right|^7 + 4\left|Z\right|^3 \ge 11$ & $\left|\left|Z\right|^7 - 4\left|Z\right|^3\right| \le 11.$

Now if $\left|Z\right| \le 1$, then $\left|Z\right|^7 + 4\left|Z\right|^3 \le 5$, hence for all Z, $\left|Z\right| > 1$

& if $\left|Z\right| \ge 2$, then $\left|\left|Z\right|^7 - 4\left|Z\right|^3\right| \le 11$ is invalid, hence for all Z, $\left|Z\right| < 2$.

All 7 roots of given equation lie in $1 < \left|Z\right| < 2$.

Q.38

Let $Z_1 = \cos A + i\sin A, Z_2 = \cos B + i\sin B$ & $Z_1 = \cos C + i\sin C$, then

$\sum \cos A = -\dfrac{3}{2}$ & $\sum \sin A = \dfrac{3\sqrt{3}}{2} \Rightarrow Z_1 + Z_2 + Z_3 = -\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i$

Now $\left|Z_1 + Z_2 + Z_3\right| \le \left|Z_1\right| + \left|Z_2\right| + \left|Z_3\right| \Rightarrow \left|Z_1 + Z_2 + Z_3\right| \le 3.$

But $\left|Z_1 + Z_2 + Z_3\right| = \sqrt{\dfrac{9}{4} + \dfrac{27}{4}} = 3.$

Hence $A = B = C = \dfrac{2\pi}{3}$ $\therefore$ $\tan A + \tan B + \tan C = -3\sqrt{3}.$

Q.39

$Z^5 - 1 = \left(Z - 1\right)\left(Z^4 + Z^3 + Z^2 + Z + 1\right)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of

$Z^4 + Z^3 + Z^2 + Z + 1 = 0$. Now by factor theorem

$\left(Z - \alpha_1\right)\left(Z - \alpha_1\right)\left(Z - \alpha_1\right)\left(Z - \alpha_1\right) = Z^4 + Z^3 + Z^2 + Z + 1$

Substituting $Z=\omega$ gives

$$\left(\omega-\alpha_1\right)\left(\omega-\alpha_1\right)\left(\omega-\alpha_1\right)\left(\omega-\alpha_1\right)=\omega^4+\omega^3+\omega^2+\omega+1=-\omega^2$$

similarly Substituting $Z=\omega^2$ gives

$$\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)=\omega^8+\omega^6+\omega^4+\omega^2+1=-\omega.$$

Hence $\dfrac{\left(\omega-\alpha_1\right)\left(\omega-\alpha_1\right)\left(\omega-\alpha_1\right)\left(\omega-\alpha_1\right)}{\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)\left(\omega^2-\alpha_1\right)}=\omega.$

Q.40

Let α be a real root of $(1+2i)\alpha^3-2(3+i)\alpha^2+(5-4i)\alpha+2a^2=0$ ...(i)

Taking conjugate gives $(1-2i)\alpha^3-2(3-i)\alpha^2+(5+4i)\alpha+2a^2=0$ ...(ii)

Adding (i) & (ii) gives $\alpha^3-6\alpha^2+5\alpha+2a^2=0$ ...(iii)

Subtracting (ii) from (i) gives $2\alpha^3-2\alpha^2-4\alpha=0$ ...(iv)

Or $2\alpha\left(\alpha^2-\alpha-2\right)=0$ gives $\alpha=0,-1,2.$

For these values of α, from (iii) we get $a=0,\pm\sqrt{6}$ & $\pm\sqrt{3}.$

Q.41

$$\left(|Z_1|+|Z_2|\right)\left|\frac{Z_1}{|Z_1|}+\frac{Z_2}{|Z_2|}\right|=\left|Z_1+Z_2+\frac{|Z_2|}{|Z_1|}Z_1+\frac{|Z_1|}{|Z_2|}Z_2\right|$$

$$\Rightarrow\left(|Z_1|+|Z_2|\right)\left|\frac{Z_1}{|Z_1|}+\frac{Z_2}{|Z_2|}\right|\le|Z_1+Z_2|+\left|\frac{|Z_2|}{|Z_1|}Z_1+\frac{|Z_1|}{|Z_2|}Z_2\right|$$

Now $\left|\dfrac{|Z_2|}{|Z_1|}Z_1+\dfrac{|Z_1|}{|Z_2|}Z_2\right|=|Z_1+Z_2|$, hence

$$\left(|Z_1|+|Z_2|\right)\left|\frac{Z_1}{|Z_1|}+\frac{Z_2}{|Z_2|}\right|\le 2|Z_1+Z_2|.$$

Q.42

$$|Z-1|=\big|Z-|Z|+|Z|-1\big|\Rightarrow|Z-1|\le\big||Z|-1\big|+\big|Z-|Z|\big|$$

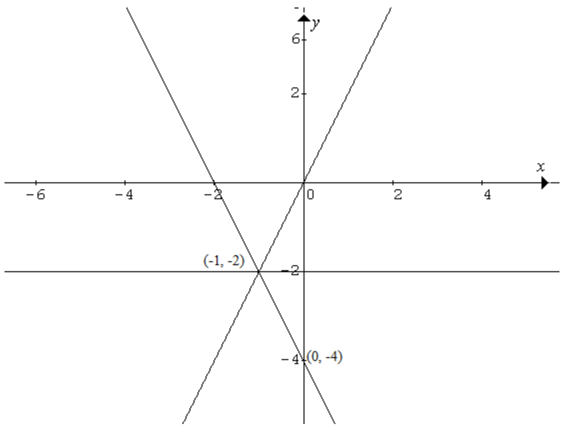
$$\Rightarrow|Z-1|\le\big||Z|-1\big|+|Z|\left|\frac{Z}{|Z|}-1\right|$$

Now let $\frac{Z}{|Z|}=\cos\theta+i\sin\theta$, then $\frac{Z}{|Z|}-1=\cos\theta-1+i\sin\theta$ or $\frac{Z}{|Z|}-1=2\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right)$.

Hence $\left|\frac{Z}{|Z|}-1\right|=\left|2\sin\frac{\theta}{2}\right|$.

Now for $|\sin\theta|\le|\theta| \Rightarrow \left|2\sin\frac{\theta}{2}\right|\le|\theta|$, therefore $|Z-1|\le\left||Z|-1\right|+|Z||\theta|$ or $|Z-1|\le\left||Z|-1\right|+|Z||\arg(Z)|$.

Q.43



The line $(2+i)Z+(2-i)\overline{Z}=0$ & $i\overline{Z}-iZ=4$,

in $x-y$ coordinates are

$(2+i)(x+iy)+(2-i)(x-iy)=0$ & $i(x+iy)-i(x-iy)=4$

i.e. $2x-y=0$ & $y=-2$.

Point of intersection : $(-1,-2)$

Also the first line passes through $(0,0)$ &

image of $(0,0)$ in $y=-2$ is $(0,-4)$.

Hence refleted line will be the line joining $(0,-4)$ & $(-1,-2)$ i.e. $y+2x+4=0$

Now put $x=\frac{Z+\overline{Z}}{2}$ & $y=\frac{Z-\overline{Z}}{2i}$ to get $\frac{Z-\overline{Z}}{2i}+2\frac{Z+\overline{Z}}{2}+4=0$

or $(2-i)Z+(2+i)\overline{Z}=8$.

Q.44

Equation of AB, $\frac{Z-a}{\overline{Z}-\overline{a}}=\frac{b-a}{\overline{b}-\overline{a}}$ & that of CD, $\frac{Z-c}{\overline{Z}-\overline{c}}=\frac{d-c}{\overline{d}-\overline{c}}$.

Eliminating $\overline{Z}$ gives $(Z-a)\frac{\overline{b}-\overline{a}}{b-a}+\overline{a}=(Z-c)\frac{\overline{d}-\overline{c}}{d-c}+\overline{c}$.

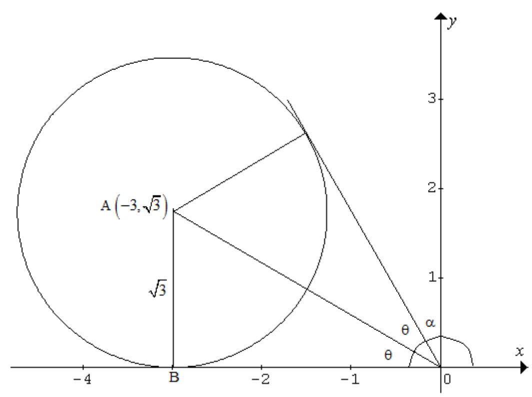
Also $|a| = |b| = |c| = |d| = r \Rightarrow \bar{a} = \frac{r}{a}, \bar{b} = \frac{r}{b}, \bar{c} = \frac{r}{c}, \bar{d} = \frac{r}{d}$, hence

$$(Z-a)\frac{\bar{b}-\bar{a}}{b-a} + \bar{a} = (Z-c)\frac{\bar{d}-\bar{c}}{d-c} + \bar{c} \Rightarrow -(Z-a)\frac{1}{ab} + \frac{1}{a} = -(Z-c)\frac{1}{cd} + \frac{1}{c}$$

or $Z = \dfrac{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}-\frac{1}{d}}{\frac{1}{ab}-\frac{1}{cd}}$.

Q.45

$\left|Z+3-\sqrt{3}i\right| = \sqrt{3}$ represents the circle having center at $\left(-3, \sqrt{3}\right)$ & radius $= \sqrt{3}$.
Let $P(Z)$ be any point on this circle, then least $\arg(z) = \theta$, where $\tan\theta$ is slope of that tangent from the origin to this circle which makes smallest angle with positive direction of real axis.



$OA = 2\sqrt{3}$ & $AB = \sqrt{3}$, hence $\theta = \alpha = \frac{\pi}{6}$

& hence least $\arg(Z) = \frac{\pi}{2} + \frac{\pi}{6}$ or $\frac{2\pi}{3}$.

Also $|Z| =$ length of tangent to this circle from origin i.e. $\left|3-\sqrt{3}i\right| - \sqrt{3}$ or 3. Hence $Z = 3e^{\frac{2i\pi}{3}}$

Q.46

$$\sin\frac{(2r-1)\pi}{14} = \sin\left(\pi - \frac{(2r-1)\pi}{14}\right) \text{ \& } \sin\frac{7\pi}{14} = 1$$

gives $\prod_{r=1}^{7}\sin\frac{(2r-1)\pi}{14} = \sin^2\frac{\pi}{14}\sin^2\frac{3\pi}{14}\sin^2\frac{5\pi}{14}$ Or $\prod_{r=1}^{7}\sin\frac{(2r-1)\pi}{14} = \cos^2\frac{\pi}{7}\cos^2\frac{2\pi}{7}\cos^2\frac{4\pi}{7}$.

$$\text{Now } \prod_{r=1}^{7} \sin\frac{(2r-1)\pi}{14} = \left(\frac{8\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}}{8\sin\frac{\pi}{7}}\right)^2$$

$$\Rightarrow \prod_{r=1}^{7} \sin\frac{(2r-1)\pi}{14} = \left(\frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}}\right)^2 = \frac{1}{64}.$$

Alternately

Let $Z = \cos\theta + i\sin\theta$, for $\theta = \frac{\pi}{7}, \frac{2\pi}{7} \& \frac{4\pi}{7}$, then $Z^7 = 1$.

Or $(Z-1)\left(Z^6 + Z^5 + Z^4 + Z^3 + Z^2 + Z + 1\right) = 0$.

Now roots of $Z^6 + Z^5 + Z^4 + Z^3 + Z^2 + Z + 1 = 0$ are $Z = e^{i\theta}$ for $\theta = \frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}$.

Dividing by Z^3 gives $Z^3 + \frac{1}{Z^3} + Z^2 + \frac{1}{Z^2} + Z + \frac{1}{Z} + 1 = 0$, roots of which are $\theta = \frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7}$.

Now $2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1 = 0$ or $8\cos^3\theta + 4\cos^2\theta - 4\cos\theta - 1 = 0$.

Hence $\cos^2\frac{\pi}{7}\cos^2\frac{2\pi}{7}\cos^2\frac{4\pi}{7} = \frac{1}{64}$.

Q.47

Let $Z = \cos\theta + i\sin\theta$, then $\sum_{r=0}^{n} \cos(r\theta) = \text{Re}\left(\sum_{r=0}^{n} Z^r\right)$

$$\text{Or } \sum_{r=0}^{n} \cos(r\theta) = \frac{\sum_{r=0}^{n} Z^r + \sum_{r=0}^{n} \overline{Z}^r}{2} \Rightarrow 2\sum_{r=0}^{n} \cos(r\theta) = \frac{Z^{n+1}-1}{Z-1} + \frac{\overline{Z}^{n+1}-1}{\overline{Z}-1}.$$

$$\Rightarrow 2\sum_{r=0}^{n} \cos(r\theta) = \frac{Z^n + \overline{Z}^n - Z^{n+1} - \overline{Z}^{n+1} - Z - \overline{Z} + 2}{2 - Z - \overline{Z}}$$

$$\Rightarrow \sum_{r=0}^{n} \cos(r\theta) = \frac{\cos n\theta - \cos(n+1)\theta - \cos\theta + 1}{2 - 2\cos\theta}.$$

$$\Rightarrow \sum_{r=0}^{n} \cos(r\theta) = \frac{\sin\frac{\theta}{2} - \sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}.$$

Q.48

Let $Z_k = \cos\theta_k + i\sin\theta_k$, then $\sum_{k=1}^{3} \cos\theta_k = 0 = \sum_{k=1}^{3} \sin\theta_k \Rightarrow Z_1 + Z_2 + Z_3 = 0.$

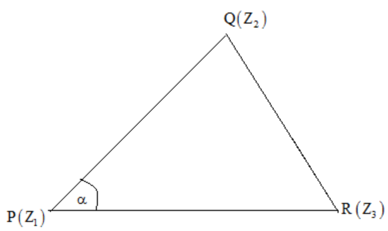
Now $Z_1 + Z_2 + Z_3 = 0 \Rightarrow Z_1^3 + Z_2^3 + Z_3^3 = 3Z_1Z_2Z_3.$

$\therefore \sum_{k=1}^{3} (\cos 3\theta_k + i\sin 3\theta_k) = 3\cos(\theta_1 + \theta_2 + \theta_3) + 3i\sin(\theta_1 + \theta_2 + \theta_3)$

Hence $\cos 3\theta_1 + \cos 3\theta_2 + \cos 3\theta_3 = 3\cos(\theta_1 + \theta_2 + \theta_3)$

& $\sin 3\theta_1 + \sin 3\theta_2 + \sin 3\theta_3 = 3\sin(\theta_1 + \theta_2 + \theta_3)$, where $\theta_1 = \alpha, \theta_2 = \beta, \theta_3 = \gamma$.

Q.49



As $\angle PQR = \angle PRQ = \frac{1}{2}(\pi - \alpha)$, hence $PQ = PR$

i.e. $|Z_2 - Z_1| = |Z_3 - Z_1|$.

Now by Coni's theorem $\frac{Z_2 - Z_1}{Z_3 - Z_1} = \cos\alpha + i\sin\alpha.$

Or $\frac{Z_2 - Z_1}{Z_3 - Z_1} - \cos\alpha = i\sin\alpha \Rightarrow \left(\frac{Z_2 - Z_1}{Z_3 - Z_1}\right)^2 - 2\left(\frac{Z_2 - Z_1}{Z_3 - Z_1}\right)\cos\alpha + \cos^2\alpha = -\sin^2\alpha$

$\Rightarrow (Z_2 - Z_1)^2 - 2(Z_2 - Z_1)(Z_3 - Z_1)\cos\alpha = (Z_3 - Z_1)^2 \Rightarrow (Z_3 - Z_2)^2 = 4(Z_3 - Z_1)(Z_1 - Z_2)\sin^2\frac{\alpha}{2}.$

Q.50

$PA = 2PB \Rightarrow |Z - 6i| = 2|Z - 3|$ or $(Z - 6i)(\overline{Z} + 6i) = 4(Z - 3)(\overline{Z} - 3)$

$\Rightarrow Z\overline{Z} = (4 + 2i)Z + (4 - 2i)\overline{Z}$ ...(i)

Now Let $Z = x + iy$, then (i) gives $x^2 + y^2 - 8x - 4y = 0$, which represents a circle.

Q.51

$|1 - Z_1\overline{Z}_2|^2 - |Z_1 - Z_2|^2 = (1 - Z_1\overline{Z}_2)(1 - \overline{Z}_1 Z_2) - (Z_1 - Z_2)(\overline{Z}_1 - \overline{Z}_2).$

$$=1+|Z_1|^2|Z_2|^2-|Z_1|^2-|Z_2|^2$$
$$=\left(1-|Z_1|^2\right)\left(1-|Z_2|^2\right).$$

Q.52

$$\frac{a-d}{b-c}+\frac{\bar{a}-\bar{d}}{\bar{b}-\bar{c}}=0 \ \& \ \frac{b-d}{c-a}+\frac{\bar{b}-\bar{d}}{\bar{c}-\bar{a}}=0 \Rightarrow \frac{c-d}{a-b}+\frac{\bar{a}-\bar{d}}{\bar{a}-\bar{b}}=0.$$

Q.53

As given α^k is n^{th} root of unity for $k=0,1,2,...,n-1$, hence

$$(Z-1)(Z-\alpha)\left(Z-\alpha^2\right)...\left(Z-\alpha^{n-1}\right)=Z^n-1$$

Or $\ln(Z-1)+\ln(Z-\alpha)+\ln\left(Z-\alpha^2\right)+...+\ln\left(Z-\alpha^{n-1}\right)=\ln\left(Z^n-1\right).$

Differentiating w.r.to Z gives

$$\frac{1}{Z-1}+\frac{1}{Z-\alpha}+\frac{1}{Z-\alpha^2}+...+\frac{1}{Z-\alpha^{n-1}}=\frac{nZ^{n-1}}{Z^n-1}.$$

Hence $\frac{1}{Z-1}+\frac{1}{Z-\alpha}+\frac{1}{Z-\alpha^2}+...+\frac{1}{Z-\alpha^{n-1}}=0 \Rightarrow Z^{n-1}=0.$

Q.54

Let $Z_1=\cos\alpha+i\sin\alpha, Z_2=\cos\beta+i\sin\beta \ \& \ Z_3=\cos\gamma+i\sin\gamma$, then

$Z_1+Z_1Z_2+Z_1Z_2Z_3=0.$

Or $Z_1(1+Z_2+Z_2Z_3)=0 \Rightarrow 1+Z_2+Z_2Z_3=0.$

$\Rightarrow \cos\beta+\cos(\beta+\gamma)=-1=\sin\beta+\sin(\beta+\gamma)$

Square and add to get

$2\cos\beta\cos(\beta+\gamma)+2\sin\beta\sin(\beta+\gamma)=-1$ or $\cos\gamma=-\frac{1}{2}.$

Hence $\beta=\gamma=\frac{2\pi}{3}$ and $\tan\frac{\beta}{2}=\sqrt{3} \ \& \tan\gamma=-\sqrt{3}.$

Q.55

Let $P=\sin 1^o \sin 2^o \sin 3^o \ ... \sin 89^o$

$\sin(90-\theta)=\cos\theta \Rightarrow P=\left(\sin 1^0 \sin 2^o \ ... \sin 44^o\right)\sin 45^o\left(\cos 44^o \ ... \cos 2^0 \cos 1^o\right)$

Also $P=\left(\sin 1^o \sin 3^o \sin 5^o ... \sin 89^o\right)\left(\sin 2^o \sin 4^o \sin 6^o ... \sin 90^0\right)$

Or $2^{44}P = \frac{1}{\sqrt{2}}\left(\sin 2^o \sin 4^o \sin 6^o \ \ldots \sin 88^o\right)$.

Hence $P = \left(\sin 1^o \sin 3^o \sin 5^o \ldots \sin 89^o\right)2^{44}\sqrt{2}\,P$ or $\sin 1^o \sin 3^o \sin 5^o \ldots \sin 89^o = \frac{1}{2^{44}\sqrt{2}}$.

Alternately

$$-1 = e^{i\pi} \Rightarrow (-1)^{1/90} = e^{\frac{(2k+1)\pi}{90}i}\ \forall\ k = 0,1,2,\ldots,89 \Rightarrow Z^{90}+1 = \prod_{k=0}^{44}\left(Z - e^{\frac{(2k+1)\pi}{90}i}\right)$$

Now multiplying conjugate pairs gives, $Z^{90}+1 = \prod_{k=0}^{45}\left(Z^2 - \left(e^{\frac{(2k+1)\pi}{90}i} + e^{\frac{(179-2k)\pi}{90}i}\right)Z + 1\right)$

$$\Rightarrow Z^{90}+1 = \prod_{k=0}^{45}\left(Z^2 - 2\cos\frac{(2k+1)\pi}{90}\ Z + 1\right) \ \Rightarrow Z^{45} + \frac{1}{Z^{45}} = \prod_{k=0}^{45}\left(Z + \frac{1}{Z} - 2\cos\frac{(2k+1)\pi}{90}\right)$$

Now for $Z = \cos 0 + i\sin 0$ we get $\prod_{k=0}^{45}\left(1 - \cos\frac{(2k+1)\pi}{90}\right) = \frac{1}{2^{44}}$

or $\prod_{k=0}^{45}\left(2\sin^2\frac{(2k+1)\pi}{180}\right) = \frac{1}{2^{44}} \Rightarrow \prod_{k=0}^{45}\left(\sin\frac{(2k+1)\pi}{180}\right) = \frac{1}{2^{44}\sqrt{2}}$.

Q.56

Let $aZ^3 + bZ^2 + cZ + d = a(Z-\alpha)(Z-\beta)(Z-\gamma)$

Now for $Z = i,\ -ai - b + ci + d = a(i-\alpha)(i-\beta)(i-\gamma)\ \ldots(i)$

and for $Z = -i,\ ai - b - ci + d = -a(i+\alpha)(i+\beta)(i+\gamma)\ \ldots(ii)$

Multiplying $(i)\,\&\,(ii)$ gives $(-ai - b + ci + d)(ai - b - ci + d) = a^2\left(1+\alpha^2\right)\left(1+\beta^2\right)\left(1+\gamma^2\right)$

or $\left(1+\alpha^2\right)\left(1+\beta^2\right)\left(1+\gamma^2\right) = \frac{(d-b)^2 + (c-a)^2}{a^2}$.

Q.57

Given $\left(1+x+x^2\right)^{3n} = \sum_{r=0}^{6n} a_r x^r$. Now recall that $1 + \omega^r + \omega^{2r} = \begin{cases} 3 & \text{if } r = 3m \\ 0 & \text{otherwise} \end{cases}$.

Substituting $1,\ \omega\,\&\,\omega^2$ in the given expansion and adding the three results gives

$$3^{3n} + \left(1+\omega+\omega^2\right)^{3n} + \left(1+\omega^2+\omega^4\right)^{3n} = \sum_{r=0}^{6n} a_r\left(1+\omega^r+\omega^{2r}\right)$$

or $3\left(a_0+a_3+a_6+\ldots+a_{6n}\right)=3^{3n}$.

Smilarly

$$3^{3n}+\left(1+\omega+\omega^2\right)^{3n}\omega^2+\left(1+\omega^2+\omega^4\right)^{3n}\omega=\sum_{r=0}^{6n}a_r\left(1+\omega^{r+2}+\omega^{2r+1}\right)$$

or $3\left(a_1+a_4+a_7+\ldots+a_{6n-2}\right)=3^{3n}$.

Q.58

$-1=e^{i\pi}\Rightarrow(-1)^{1/8}=e^{\frac{(2k+1)\pi}{8}i}\ \forall\ k=0,1,2,\ldots,7$. Hence

$Z^8=-1\Rightarrow\left(Z-e^{\frac{\pi}{8}i}\right)\left(Z-e^{\frac{3\pi}{8}i}\right)\ldots\left(Z-e^{\frac{15\pi}{8}i}\right)$, where $\left\{e^{\frac{\pi}{8}i},e^{\frac{15\pi}{8}i}\right\},\left\{e^{\frac{3\pi}{8}i},e^{\frac{13\pi}{8}i}\right\}$...etc. are conjugate pairs.

$$Z^8+1=\left(Z^2-\left(e^{\frac{\pi}{8}i}+e^{\frac{15\pi}{8}i}\right)Z+1\right)\left(Z^2-\left(e^{\frac{3\pi}{8}i}+e^{\frac{13\pi}{8}i}\right)Z+1\right)\left(Z^2-\left(e^{\frac{5\pi}{8}i}+e^{\frac{11\pi}{8}i}\right)Z+1\right)\left(Z^2-\left(e^{\frac{7\pi}{8}i}+e^{\frac{9\pi}{8}i}\right)Z+1\right)$$

$$Z^8+1=\left(Z^2-2\cos\frac{\pi}{8}Z+1\right)\left(Z^2-2\cos\frac{3\pi}{8}Z+1\right)\left(Z^2-2\cos\frac{5\pi}{8}Z+1\right)\left(Z^2-2\cos\frac{7\pi}{8}Z+1\right)$$

Further the above result implies

$$Z^4+\frac{1}{Z^4}=\left(Z+\frac{1}{Z}-2\cos\frac{\pi}{8}\right)\left(Z+\frac{1}{Z}-2\cos\frac{3\pi}{8}\right)\left(Z+\frac{1}{Z}-2\cos\frac{5\pi}{8}\right)\left(Z+\frac{1}{Z}-2\cos\frac{7\pi}{8}\right)$$

Now for $Z=\cos\theta+i\sin\theta$ we get $\cos4\theta=8\left(\cos\theta-\cos\frac{\pi}{8}\right)\left(\cos\theta-\cos\frac{3\pi}{8}\right)\left(\cos\theta-\cos\frac{5\pi}{8}\right)\left(\cos\theta-\cos\frac{7\pi}{8}\right)$.

Now for $\theta=0$, $\left(1-\cos\frac{\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{5\pi}{8}\right)\left(1-\cos\frac{7\pi}{8}\right)=\frac{1}{8}$

or $\left(2\sin^2\frac{\pi}{16}\right)\left(2\sin^2\frac{3\pi}{16}\right)\left(2\sin^2\frac{5\pi}{16}\right)\left(2\sin^2\frac{7\pi}{16}\right)=\frac{1}{8}$

Hence $\sin^2\frac{\pi}{16}\sin^2\frac{3\pi}{16}\sin^2\frac{5\pi}{16}\sin^2\frac{7\pi}{16}=\frac{1}{128}$.

Q.59

(a) $\frac{Z_2-Z_1}{Z_3-Z_1}=\cos\frac{\pi}{3}+i\cos\frac{\pi}{3}\Rightarrow\left(\frac{Z_2-Z_1}{Z_3-Z_1}-\frac{1}{2}\right)^2=-\frac{3}{4}$

or $\left(\frac{Z_2-Z_1}{Z_3-Z_1}\right)^2-2\left(\frac{Z_2-Z_1}{Z_3-Z_1}\right)+1=0$

$\therefore Z_1^{\,2}+Z_2^{\,2}+Z_3^{\,2}=Z_1Z_2+Z_2Z_3+Z_3Z_1$.

$(b)\ Z_1+Z_2+Z_3=3Z_0 \Rightarrow Z_1^2+Z_2^2+Z_3^2+2Z_1Z_2+2Z_2Z_3+2Z_3Z_1=9Z_0^2$

Also for an equilateral Δ, $Z_1^2+Z_2^2+Z_3^2=Z_1Z_2+Z_2Z_3+Z_3Z_1$.

Hence $Z_1^2+Z_2^2+Z_3^2=3Z_0^2$.

$(c)\ Z_1+Z_2+\ldots+Z_{3n}=3nZ_0\ \ \&\ \ (Z_1-Z_0)^2+(Z_2-Z_0)^2+\ldots+(Z_{3n}-Z_0)^2=0$

$\Rightarrow \left(Z_1^2+Z_2^2+\ldots+Z_{3n}^2\right)-2Z_0\left(Z_1+Z_2+\ldots+Z_{3n}\right)+3nZ_0^2=0$

or $Z_1^2+Z_2^2+\ldots+Z_{3n}^2=3nZ_0^2$

Q.60

(a) Using $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ we get

$\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{6\pi}{7}=4\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}.$

Now $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{2}.$

(b) Let center be the origin and A_1 be on real $-$ axis i.e. $OA_1=1$

Now each of the internal angle will be $\frac{2\pi}{7}$, hence $OA_2=e^{\frac{2i\pi}{7}}, OA_3=e^{\frac{4i\pi}{7}}\ \&\ OA_4=e^{\frac{6i\pi}{7}}$.

$\Rightarrow A_1A_2=OA_2-OA_1=e^{\frac{2i\pi}{7}}-1, A_1A_3=e^{\frac{4i\pi}{7}}-1\ \&\ A_1A_4=OA_4-OA_1=e^{\frac{6i\pi}{7}}-1.$

Hence $|A_1A_2|+|A_1A_4|-|A_1A_3|=\left|e^{\frac{2i\pi}{7}}-1\right|+\left|e^{\frac{4i\pi}{7}}-1\right|-\left|e^{\frac{6i\pi}{7}}-1\right|.$

$\Rightarrow 2\sin\frac{\pi}{7}+2\sin\frac{2\pi}{7}-2\sin\frac{3\pi}{7}=\sqrt{7}.$