Applications Of Derivatives

Exercise 1(A)

Q.1 (b)

Given $x^2 + y^2 = 2c^2$

Differentiating *w.r.t. x*, $2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(c,c)} = -1$

Q.2 (a)

Given curve $x^2 = 3 - 2y$

diff. *w.r.t. x*, $2x = -\frac{2dy}{dx}$; $\frac{dy}{dx} = -x$

Slope of the line = – 1

$\frac{dy}{dx} = -x = -1$; $x = 1$

$\therefore y = 1$ point (1, 1)

Q.3 (b)

Given $y = 2x^2 - x + 1$

Let the co-ordinate of *P* is (*h*, *k*) then $\left(\frac{dy}{dx}\right)_{(h,k)} = 4h - 1$

Clearly 4h – 1 = 3.

h = 1$\Rightarrow$ k = 3. *P* is (1, 2).

Q.4 (d)

$x^2 = -4y \Rightarrow 2x = -4\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{2} \Rightarrow \left(\frac{dy}{dx}\right)_{(-4,-4)} = 2$.

We know that equation of tangent is

$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$

$\Rightarrow y + 4 = 2(x + 4)$

$\Rightarrow 2x - y + 4 = 0$.

Q.5 (b)

$y = \sin\frac{\pi x}{2} \Rightarrow \frac{dy}{dx} = \frac{\pi}{2}\cos\frac{\pi}{2}x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 0$

$\therefore$ Equation of normal is $y - 1 = \frac{1}{0}(x - 1)$

$\Rightarrow x = 1$.

Q.6 (d)

Curve is $y = be^{-x/a}$

Since the curve crosses *y*-axis (*i.e.*, $x = 0$) $\therefore$ $y = b$

Now $\frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$.

At point (0, *b*), $\left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a}$

$\therefore$ Equation of tangent is $y - b = \frac{-b}{a}(x - 0)$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$.

Q.7 (d)

Slope of the normal $= \frac{-1}{dy/dx}$

$\Rightarrow \tan\frac{3\pi}{4} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(3,4)}}$

$\therefore \left(\frac{dy}{dx}\right)_{(3,4)} = 1$; $f'(3) = 1$.

Q.8 (d)

$y^3 + 3x^2 = 12y \Rightarrow 3y^2 . \frac{dy}{dx} + 6x = 12 . \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(3y^2 - 12) + 6x = 0$

$\Rightarrow \frac{dy}{dx} = \frac{6x}{12 - 3y^2}$

$\Rightarrow \frac{dx}{dy} = \frac{12 - 3y^2}{6x}$

Tangent is parallel to *y*-axis, $\frac{dx}{dy} = 0$

$\Rightarrow 12 - 3y^2 = 0$ or $y = \pm 2$.

Then $x = \pm\frac{4}{\sqrt{3}}$, for $y = 2$

$y = -2$ does not satisfy the equation of the curve,

$\therefore$ The point is $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

Q.9 (c)

Let the point be (x_1, y_1)

$\therefore y_1 = be^{-x_1/a}$ ......(i)

Also, curve $y = be^{-x/a} \Rightarrow \frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-b}{a}e^{-x_1/a} = \frac{-y_1}{a} \qquad \text{(by (i))}$$

Now, the equation of tangent of given curve at point (x_1, y_1) is

$$y - y_1 = \frac{-y_1}{a}(x - x_1) \Rightarrow \frac{x}{a} + \frac{y}{y_1} = \frac{x_1}{a} + 1$$

Comparing with $\frac{x}{a} + \frac{y}{b} = 1$, we get, $y_1 = b$ and $1 + \frac{x_1}{a} = 1 \Rightarrow x_1 = 0$

Hence, the point is (0, *b*).

Q.10 (d)

$$y = x^3 - 3x^2 - 9x + 5 \Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 9\,.$$

We know that this equation gives the slope of the tangent to the curve.

The tangent is parallel to *x*-axis $\frac{dy}{dx} = 0$

Therefore, $3x^2 - 6x - 9 = 0$

$\Rightarrow x = -1, 3$.

Q.11 (b)

Given curve $y^2 = x$ and $x^2 = y$

Differentiating *w.r.t. x*, $2y\frac{dy}{dx} = 1$ and $2x = \frac{dy}{dx}$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{1}{2} \text{ and } \left(\frac{dy}{dx}\right)_{(1,1)} = 2$$

Angle between the curve

$$\Rightarrow \tan\phi = \frac{2 - \frac{1}{2}}{1 + \frac{1}{2}.2}$$

$$\Rightarrow \tan\phi = \frac{3}{4} \Rightarrow \phi = \tan^{-1}\frac{3}{4}\,.$$

Q.12 (a)

Clearly the point of intersection of curves is (0, 1)

Now, slope of tangent of first curve, $m_1 = \frac{dy}{dx} = a^x \log a \Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = m_1 = \log a$

Slope of tangent of second curve, $m_2 = \frac{dy}{dx} = b^x \log b \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(0,1)} = \log b$

$$\therefore \tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\log a - \log b}{1 + \log a \log b}\,.$$

Q.13 (b)

The equation of two curves are $xy = 6$ and $x^2 y = 12$ from (i) we obtain $y = \frac{6}{x}$

putting this value of y in equation (ii) to obtain $x^2\left(\frac{6}{x}\right) = 12 \Rightarrow 6x = 12 \Rightarrow x = 2$

Putting $x = 2$ in (i) or (ii) we get, $y = 3$.

Thus, the two curves intersect at *P*(2, 3)

Differentiating (i) *w.r.t.* x, we get $x\frac{dy}{dx}+y=0 \Rightarrow \frac{dy}{dx}=\frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)}=-\frac{3}{2}=m_1$

Differentiating (ii) *w.r.t.* x, we get $x^2\frac{dy}{dx}+2xy=0 \Rightarrow \frac{dy}{dx}=\frac{-2y}{x}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)}=-3=m_2$$

$$\Rightarrow \tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\left(\frac{-3}{2}+3\right)}{\left(1+\left(\frac{-3}{2}\right)(-3)\right)}=\frac{3}{11}$$

$$\Rightarrow \theta=\tan^{-1}\frac{3}{11}.$$

Q.14 (c)

Equation of the curve $x^2y^2=a^4$.

Differentiating the given equation,

$$x^2 2y\frac{dy}{dx}+y^2 2x=0 \Rightarrow \frac{dy}{dx}=\frac{-y}{x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(-a,a)}=-\left(\frac{a}{-a}\right)=1$$

Therefore, sub-tangent = $\frac{y}{\left(\frac{dy}{dx}\right)}=a$.

Q.15 (a)

$$y^n=a^{n-1}x \Rightarrow ny^{n-1}\frac{dy}{dx}=a^{n-1}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)=\frac{a^{n-1}}{ny^{n-1}}$$

$\therefore$ Length of the subnormal = $y\frac{dy}{dx}=\frac{ya^{n-1}}{ny^{n-1}}=\frac{a^{n-1}y^{2-n}}{n}$

We also know that if the subnormal is constant,

then $\frac{a^{n-1}}{n}.y^{2-n}$ should not contain y.

Therefore, $2-n=0$ or $n=2$.

Q.16 (a)

$$\sqrt{x}+\sqrt{y}=\sqrt{a} \Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \therefore \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

Hence tangent at (x, y) is $Y-y=-\frac{\sqrt{y}}{\sqrt{x}}(X-x)$

or $X\sqrt{y}+Y\sqrt{x}=\sqrt{xy}(\sqrt{x}+\sqrt{y})=\sqrt{axy}$

or $\frac{X}{\sqrt{a}\sqrt{x}} + \frac{Y}{\sqrt{a}\sqrt{y}} = 1$.

Clearly its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

Sum of the intercepts = $\sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{a}.\sqrt{a} = a$.

Q.17 (c)

Differentiating the given equation *w.r.t.* x , $2y\frac{dy}{dx} = 4$

at point (2, 4) $\frac{dy}{dx} = \frac{1}{2}$

$$P = \frac{y_1 - x_1\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$= \frac{4 - 2\left(\frac{1}{2}\right)}{\sqrt{1 + \frac{1}{4}}} = \frac{6}{\sqrt{5}} .$$

Q.18 (b)

$f(x) = 3x + \frac{2}{x} \Rightarrow f'(x) = 3 - \frac{2}{x^2}$

Clearly $f'(x) > 0$ on the interval (1, 3)

$\therefore$ $f(x)$ is strictly increasing.

Q.19 (c)

$f(x) = (x-1)^2 - 1$

Hence decreasing in $x < 1$

Alternative method:

$f'(x) = 2x - 2 = 2(x-1)$

To be decreasing, $2(x-1) < 0$

$\Rightarrow (x-1) < 0 \Rightarrow x < 1$.

Q.20 (a)

$f'(x) = 6x^2 + 36x - 96 > 0$, for increasing

$\Rightarrow f'(x) = 6(x+8)(x-2) \geq 0$

$\Rightarrow x \geq 2, x \leq -8$.

Q.21 (a)

Let $y = x^x \Rightarrow \frac{dy}{dx} = x^x(1 + \log x)$;

For $\frac{dy}{dx} > 0$, $x^x(1 + \log x) > 0$

$\Rightarrow 1 + \log x > 0$

$\Rightarrow \log_e x > \log_e \frac{1}{e}$

For this to be positive, x should be greater than $\frac{1}{e}$.

Q.22 (b)

$f(x)$ will be monotonically decreasing, if $f'(x) < 0$.

$\Rightarrow f'(x) = -\sin x - 2p < 0$

$\Rightarrow \frac{1}{2}\sin x + p > 0 \Rightarrow p > \frac{1}{2}$ $[\because -1 \le \sin x \le 1]$

Q.23 (c)

$f'(x) = 5x^4 - 60x^2 + 240$

$= 5(x^4 - 12x^2 + 48) = 5[(x^2 - 6)^2 + 12]$

$\Rightarrow f'(x) > 0 \ \forall\ x \in R$

i.e., $f(x)$ is monotonically increasing everywhere.

Q.24 (d)

If $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$,

then $f'(x) \le 0$ for all $x \in R$

$\Rightarrow 3(a+2)x^2 - 6ax + 9a \le 0$ for all $x \in R$

$\Rightarrow (a+2)x^2 - 2ax + 3a \le 0$ for all $x \in R$

$\Rightarrow a + 2 < 0$ and discriminant ≤ 0

$\Rightarrow a < -2$ and $-8a^2 - 24a \le 0$

$\Rightarrow a < -2$ and $a(a+3) \ge 0 \Rightarrow a < -2$

and $a \le -3$ or $a \ge 0 \Rightarrow a \le -3$

$\Rightarrow -\infty < a \le -3$

Q.25 (d)

The function is monotonic increasing if, $f'(x) > 0$

$$\Rightarrow \frac{(2\sin x + 3\cos x)(\lambda\cos x - 6\sin x)}{(2\sin x + 3\cos x)^2} - \frac{(\lambda\sin x + 6\cos x)(2\cos x - 3\sin x)}{(2\sin x + 3\cos x)^2} > 0$$

$\Rightarrow 3\lambda(\sin^2 x + \cos^2 x) - 12(\sin^2 x + \cos^2 x) > 0$

$\Rightarrow 3\lambda - 12 > 0 \Rightarrow \lambda > 4$.

Q.26 (b)

Let $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$

$$\therefore \quad f'(x) = \frac{\ln(e+x) \times \frac{1}{\pi + x} - \ln(\pi + x)\frac{1}{e + x}}{\{\ln(e+x)\}^2}$$

$$= \frac{(e+x)\ln(e+x) - (\pi + x)\ln(\pi + x)}{\{\ln(e+x)\}^2 \times (e+x)(\pi + x)}$$

$\Rightarrow f'(x) < 0$ for all $x \ge 0$ $\{\because \pi > e\}$.

Hence, $f(x)$ is decreasing in $[0, \infty)$.

Q.27 (c)

Obviously, here $\cos 3x$ in not decreasing in $\left(0, \frac{\pi}{2}\right)$ because $\frac{d}{dx}\cos 3x = -3\sin 3x$.

But at $x = 75^o$, $-3\sin 3x > 0$.

Hence the result.

Q.28 (b)

We have $f(x) = x - e^x + \tan\left(\frac{2\pi}{7}\right) \Rightarrow f'(x) = 1 - e^x$

For $f(x)$ to be increasing, we must have $f'(x) > 0$

$\Rightarrow 1 - e^x > 0 \Rightarrow e^x < 1$

$\Rightarrow x < 0 \Rightarrow x \in (-\infty, 0)$

Q.29 (a)

$f'(x) = e^{x(1-x)} + x \,.\, e^{x(1-x)}.(1 - 2x)$

$= e^{x(1-x)}\{1 + x(1 - 2x)\}$

$= e^{x(1-x)}.(-2x^2 + x + 1)$

Now by the sign-scheme for $-2x^2 + x + 1$

$f'(x) \ge 0$, if $x \in \left[-\frac{1}{2}, 1\right]$, because $e^{x(1-x)}$ is always positive.

So, $f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$.

Q.30 (b)

$f(x) = x\sin x + \cos x + \cos^2 x$

$\therefore \; f'(x) = \sin x + x\cos x - \sin x - 2\cos x \sin x$

$= \cos x(x - 2\sin x)$

Hence $x \to 0$ to π, then $f'(x) \le 0$, *i.e.*,

$f(x)$ is decreasing function.

Q.31 (b)

Let $f(x) = x^2 e^{-x}$

$\Rightarrow \frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2)$

Hence $f'(x) \ge 0$ for every $x \in [0, 2]$,

therefore it is non-decreasing in [0, 2].

Q.32 (b)

$f(x) = \sin^4 x + \cos^4 x$

$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$

$= 1 - \frac{4\sin^2 x \cos^2 x}{2} = 1 - \frac{\sin^2 2x}{2} = 1 - \frac{1}{4}(2\sin^2 2x)$

$= 1 - \left(\frac{1 - \cos 4x}{4}\right) = \frac{3}{4} + \frac{1}{4}\cos 4x$

Hence function $f(x)$ is increasing when $f'(x) > 0$

$f'(x) = -\sin 4x > 0 \Rightarrow \sin 4x < 0$

Hence $\pi < 4x < \frac{3\pi}{2}$ or $\frac{\pi}{4} < x < \frac{3\pi}{8}$.

Q.33 (c)

From mean value theorem $f'(c) = \frac{f(b) - f(a)}{b - a}$

$a = 0, f(a) = 0 \Rightarrow b = \frac{1}{2}, f(b) = \frac{3}{8}$

$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$,

$f'(c) = (c-1)(c-2) + c(c-2) + c(c-1)$

$= c^2 - 3c + 2 + c^2 - 2c + c^2 - c$, $f'(c) = 3c^2 - 6c + 2$

According to mean value theorem

$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8}\right) - 0}{\left(\frac{1}{2}\right) - 0} = \frac{3}{4} \Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$$

$$c = \frac{6 \pm \sqrt{36 - 15}}{2 \times 3} = \frac{6 \pm \sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}.$$

Q.34 (a)

$f'(x_1) = \frac{-1}{x_1^2}$,

$$\therefore \frac{-1}{x_1^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab} \Rightarrow x_1 = \sqrt{ab}.$$

Q.35 (a)

Given that equation of curve $y = x^3 = f(x)$

So $f(2) = 8$ and $f(-2) = -8$

Now $f'(x) = 3x^2 \Rightarrow f'(x) = \frac{f(2) - f(-2)}{2 - (-2)}$

$$\Rightarrow \frac{8 - (-8)}{4} = 3x^2; \therefore x = \pm \frac{2}{\sqrt{3}}.$$

Q.36 (c)

To determine 'c' in Rolle's theorem, $f'(c) = 0$

Here $f'(x) = (x^2 + 3x)e^{-(1/2)x}.\left(-\frac{1}{2}\right) + (2x + 3)e^{-(1/2)x}$

$= e^{-(1/2)x}\left\{-\frac{1}{2}(x^2 + 3x) + 2x + 3\right\}$

$= -\frac{1}{2}e^{-(x/2)}\{x^2 - x - 6\}$

$\therefore f'(c) = 0 \Rightarrow c^2 - c - 6 = 0 \Rightarrow c = 3, -2$.

But $c = 3 \notin [-3, 0]$, Hence $c = -2$.

Q.37 (a)

$f(x)=x^3-6x^2+ax+b \Rightarrow f'(x)=3x^2-12x+a$

$\Rightarrow f'(c)=0 \Rightarrow f'\left(2+\frac{1}{\sqrt{3}}\right)=0$

$\Rightarrow 3\left(2+\frac{1}{\sqrt{3}}\right)^2-12\left(2+\frac{1}{\sqrt{3}}\right)+a=0$

$\Rightarrow 3\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right)-12\left(2+\frac{1}{\sqrt{3}}\right)+a=0$

$\Rightarrow 12+1+4\sqrt{3}-24-4\sqrt{3}+a=0$

Q.38 (a)

$y=x^5-5x^4+5x^3-10$

$\therefore \frac{dy}{dx}=5x^4-20x^3+15x^2$

$=5x^2(x^2-4x+3)$

$=5x^2(x-3)(x-1)$

$\frac{dy}{dx}=0$, gives $x=0,1,3$ ......(i)

Now, $\frac{d^2y}{dx^2}=20x^3-60x^2+30x=10x(2x^2-6x+3)$

and $\frac{d^3y}{dx^3}=10(6x^2-12x+3)$

For $x=0: \frac{dy}{dx}=0, \frac{d^2y}{dx^2}=0, \frac{d^3y}{dx^3}\neq 0$, $\therefore$ Neither minimum nor maximum

For $x=1$, $\frac{d^2y}{dx^2}=-10$ =negative, $\therefore$ Maximum value $y_{\text{max.}}=-9$

For $x=3$, $\frac{d^2y}{dx^2}=90$ =positive, $\therefore$ Minimum value $y_{\text{min.}}=-37$.

Q.39 (c)

$y=\sin x(1+\cos x)=\sin x+\frac{1}{2}\sin 2x$

$\therefore \frac{dy}{dx}=\cos x+\cos 2x$ and $\frac{d^2y}{dx^2}=-\sin x-2\sin 2x$

On putting $\frac{dy}{dx}=0, \cos x+\cos 2x=0$

$\Rightarrow \cos x=-\cos 2x=\cos(\pi-2x)$

$\Rightarrow x=\pi-2x$

$\therefore x=\frac{\pi}{3}$, $\therefore \left(\frac{d^2y}{dx^2}\right)_{x=\pi/3}=-\sin\left(\frac{1}{3}\pi\right)-2\sin\left(\frac{2}{3}\pi\right)$

$=\frac{-\sqrt{3}}{2}-2.\frac{\sqrt{3}}{2}=\frac{-3\sqrt{3}}{2}$ which is negative.

$\therefore$ at $x=\frac{\pi}{3}$ the function is maximum.

Q.40 (d) $\frac{dy}{dx} = \frac{a}{x} + 2bx + 1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0$

$\Rightarrow a = -2b - 1$

and $\left(\frac{dy}{dx}\right)_{x=2} = \frac{a}{2} + 4b + 1 = 0 \Rightarrow \frac{-2b-1}{2} + 4b + 1 = 0$

$\Rightarrow -b + 4b + \frac{1}{2} = 0$

$\Rightarrow 3b = \frac{-1}{2} \Rightarrow b = \frac{-1}{6}$ and $a = \frac{1}{3} - 1 = \frac{-2}{3}$.

Q.41 (b)

$f(x) = \left(\frac{1}{x}\right)^x \Rightarrow f'(x) = \left(\frac{1}{x}\right)^x \left(\log\frac{1}{x} - 1\right)$

$f'(x) = 0 \Rightarrow \log\frac{1}{x} = 1 = \log e$

$\Rightarrow \frac{1}{x} = e \Rightarrow x = \frac{1}{e}$.

Therefore, maximum value of function is $e^{1/e}$.

Q.42 (b)

$y = f(x) = -x^3 + 3x^2 + 9x - 27$

The slope of this curve $f'(x) = -3x^2 + 6x + 9$

Let $g(x) = f'(x) = -3x^2 + 6x + 9$

Differentiate with respect to x, $g'(x) = -6x + 6$

Put $g'(x) = 0 \Rightarrow x = 1$

Now, $g''(x) = -6 < 0$ and hence at $x = 1, g(x)$

(Slope) will have maximum value.

$\therefore [g(1)]_{\text{max.}} = -3 \times 1 + 6 + 9 = 12$.

Q.43 (c)

$f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\, dt$, $\therefore$ $f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$

For local minima, slope *i.e.*, $f'(x)$ should change sign from – *ve* to +*ve*

$f'(x) = 0 \Rightarrow x = 0, 1, 2, 3$

If $x = 0 - h$, where h is a very small number, then $f'(x) = (-)(-)(-1)(-1)(-1) = -ve$

If $x = 0 + h$, $f'(x) = (+)(+)(-)(-1)(-1) = -ve$

Hence at $x = 0$ neither maxima nor minima.

If $x = 1 - h$, $f'(x) = (+)(+)(-)(-1)(-1) = -ve$

If $x = 1 + h$, $f'(x) = (+)(+)(+)(-1)(-1) = +ve$

Hence, at $x = 1$ there is a local minima.

If $x = 2 - h$, $f'(x) = (+)(+1)(+)(-)(-) = +ve$

If $x = 2 + h$, $f'(x) = (+)(+)(+)(+)(-1) = -ve$

Hence at $x = 2$ there is a local maxima.

If $x = 3 - h$, $f'(x) = (+)(+)(+)(+)(-) = -ve$

If $x = 3 + h$, $f'(x) = (+)(+)(+)(+)(+) = +ve$

Hence at $x = 3$ there is a local minima.

Q.44 (c)

$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$f'(x) = 6x^2 - 18ax + 12a^2$

$f''(x) = 12x - 18a$

For maximum and minimum, $6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow x^2 - 3ax + 2a^2 = 0$

$x = a$ or $x = 2a$ at $x = a$ maximum and at $x = 2a$ minimum

$\because p^2 = q$

$a^2 = 2a \Rightarrow a = 2$ or $a = 0$ but $a > 0$, therefore $a = 2$.

Q.45 (c)

$\phi(x) = \int_1^x e^{-t^2/2}(1-t^2)dt \Rightarrow \phi'(x) = e^{-x^2/2}(1-x^2)$

Now $\phi'(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

Hence, $x = \pm 1$ are points of extrema of $\phi(x)$.

Q.46 (c)

Let $y = x^3 - 18x^2 + 96x \Rightarrow \frac{dy}{dx} = 3x^2 - 36x + 96 = 0$

$\therefore x^2 - 12x + 32 = 0 \Rightarrow (x-4)(x-8) = 0, x = 4, 8$

Now, $\frac{d^2y}{dx^2} = 6x - 36$ at $x = 4, \frac{d^2y}{dx^2} = 24 - 36 = -12 < 0$

$\therefore$ at $x = 4$ function will be maximum and $[f(x)]_{\text{max.}} = 64 - 288 + 384 = 160$

at $x = 8\ \frac{d^2y}{dx^2} = 48 - 36 = 12 > 0$

$\therefore$ at $x = 8$ function will be minimum and $[f(x)]_{\text{min.}} = 128$.

Q.47 (d)

$y = 2\cos 2x - \cos 4x$

$= 2\cos 2x(1 - \cos 2x) + 1$

$= 4\cos 2x \sin^2 x + 1$

Obviously, $\sin^2 x \geq 0$

Therefore, to be least value of y, cos $2x$ should be least *i.e.*, – 1.

Hence least value of y is – 4 + 1 = –3.

Q.48 (a)

$f(x) = x^2 \log x \Rightarrow f'(x) = (2\log x + 1)x$

Now $f'(x) = 0 \Rightarrow x = e^{-1/2}, 0$

$\because 0 < e^{-1/2} < 1$, $\because$ None of these critical points lies in the interval $[1, e]$

$\therefore$ So we only compare the value of $f(x)$ at the end points 1 and e.

We have $f(1) = 0, f(e) = e^2$

$\therefore$ greatest value = e^2

Q.49 (a)

$xy=1 \Rightarrow y=\frac{1}{x}$ and let $z=x+y$

$z=x+\frac{1}{x} \Rightarrow \frac{dz}{dx}=1-\frac{1}{x^2}$

$\Rightarrow \frac{dz}{dx}=0 \Rightarrow 1-\frac{1}{x^2}=0$

$\Rightarrow x=-1,+1$ and $\frac{d^2z}{dx^2}=\frac{2}{x^3}$

$\left(\frac{d^2z}{dx^2}\right)_{x=1}=\frac{2}{1}=2=+ve$,

$\therefore$ $x=1$ is point of minima.

$x=1, y=1$,

$\therefore$ minimum value = $x+y=2$.

Q.50 (c)

Let $x+y=4$ or $y=4-x$

$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$ or $f(x)=\frac{4}{xy}=\frac{4}{x(4-x)}$

$f(x)=\frac{4}{4x-x^2}$, $f'(x)=\frac{-4}{(4x-x^2)^2}.(4-2x)$

Put $f'(x)=0 \Rightarrow 4-2x=0 \Rightarrow x=2$ and $y=2$

$\therefore$ min. $\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{1}{2}+\frac{1}{2}=1$.

Q.51 (b)

Let number = x, then cube = x^3

Now $f(x)=x-x^3$ (Maximum) $\Rightarrow f'(x)=1-3x^2$

Put $f'(x)=0 \Rightarrow 1-3x^2=0 \Rightarrow x=\pm\frac{1}{\sqrt{3}}$

Because $f''(x)=-6x=-ve$. when $x=+\frac{1}{\sqrt{3}}$.

Q.52 (c)

$2x+2y=100 \Rightarrow x+y=50$ .....(i)

Let area of rectangle is A, then $A=xy \Rightarrow y=\frac{A}{x}$

From (i), $x+\frac{A}{x}=50 \Rightarrow A=50x-x^2$

$\Rightarrow \frac{dA}{dx}=50-2x$

for maximum area $\frac{dA}{dx}=0$

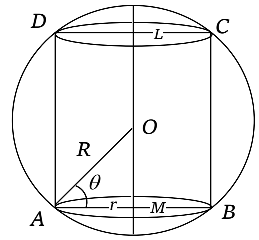
$\therefore$ $50-2x=0 \Rightarrow x=25$ and $y=25$

$\therefore$ adjacent sides are 25 *cm* and 25 *cm*.

Q.53 (b)

If r be the radius and h the height, the from the figure,

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h^2 = 4(R^2 - r^2)$$



Now, $V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$

$$\therefore \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + 2\pi r^2 . \frac{1}{2} \frac{(-2r)}{\sqrt{R^2 - r^2}}$$

For max. or min., $\frac{dV}{dr} = 0$

$$\Rightarrow 4\pi r\sqrt{R^2 - r^2} = \frac{2\pi r^3}{\sqrt{R^2 - r^2}} \Rightarrow 2(R^2 - r^2) = r^2$$

$$\Rightarrow 2R^2 = 3r^2 \Rightarrow r = \sqrt{\frac{2}{3}}R \Rightarrow \frac{d^2V}{dr^2} = -ve .$$

Hence V is max. when $r = \sqrt{\frac{2}{3}}R$.

Q.54 (a)

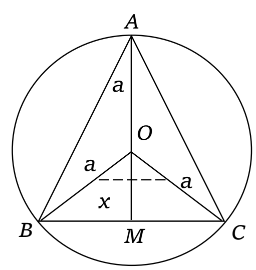
Let $OM = x$

Then height of cone *i.e.*, $h = x + a$ (where a is radius of sphere)

Radius of base of cone = $\sqrt{a^2 - x^2}$

Therefore, volume $V = \frac{1}{3}\pi(a^2 - x^2)(x + a) \Rightarrow \frac{dV}{dx} = \frac{\pi}{3}(a + x)(a - 3x)$

Now, $\frac{dV}{dx} = 0 \Rightarrow x = -a, \frac{a}{3}$



But $x \neq -a$, So, $x = \frac{a}{3}$

The volume is maximum at $x = \frac{a}{3}$

Height of a cone $h = a + \frac{a}{3} = \frac{4}{3}a$

Therefore ratio of height and diameter = $\frac{\frac{4}{3}a}{2a} = \frac{2}{3}$.

Applications Of Derivatives
EXERCISE 1(B)

1. $y = 2x^3 + 13x^2 + 5x + 9$

$\therefore \quad \frac{dy}{dx} = 6x^2 + 26x + 5$

Let the point be (h, k)

$\therefore \quad k = 2h^3 + 13h^2 + 5h + 9$

$\therefore \quad \frac{y - \left(2h^3 + 13h^2 + 5h + 9\right)}{x - h} = 6h^2 + 26h + 5$

substituting $(0, 0)$

$\therefore \quad 2h^3 + 13h^2 + 5h + 9 = 6h^3 + 26h^2 + 5h$

$\therefore \quad 4h^3 + 13h^2 - 9 = 0$

$\Rightarrow \quad h = -1, k = 15.$

2. $x = a\left(t + \sin t \cos t\right)$

$y = a\left(1 + \sin t\right)^2$

$\therefore \quad \frac{dx}{dt} = a\left(1 + \cos 2t\right)$

$\frac{dy}{dt} = 2a\left(1 + \sin t\right)\cos t$

$\therefore \quad \frac{dy}{dx} = \frac{2\cos t + \sin 2t}{1 + \cos 2t}$

$$= \frac{2\cos t\left(1 + \sin t\right)}{2\cos^2 t} = \frac{\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right)^2}{\cos^2\frac{t}{2} - \sin^2\frac{t}{2}}$$

$$= \frac{1 + \tan\frac{t}{2}}{1 - \tan\frac{t}{2}} \qquad = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

3. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$\therefore \quad \frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}$

At (a, b)

$$\frac{dy}{dx} = -\frac{b}{a}$$

$\therefore \quad b(x-a) + a(y-b) = 0$

$\therefore \quad bx + ay = 2ab$

$\therefore \quad \frac{x}{a} + \frac{y}{b} = 2.$

4. $y = be^{-x/a}$

$x = 0 \quad \Rightarrow \quad y = b$

$y' = -\frac{b}{a}e^{-x/a}$

$\therefore \quad \left.\frac{dy}{dx}\right|_{(0,b)} = -\frac{b}{a}$

$\therefore \quad a(y-b) = -bx$

$\therefore \quad bx + y = ab$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1.$

5. $y = 3x^2 + bx + 2$

$x = 0 \quad \Rightarrow \quad y = 2$

$\frac{dy}{dx} = 6x + b$

$\therefore \quad \left.\frac{dy}{dx}\right|_{x=0} = b = 4$ .... Given

$\therefore \quad b = 4$

6. $y = \frac{8 - x^2}{2}$

$\therefore \quad \frac{dy}{dx} = -x = -2$ [Given]

$\therefore \quad x = 2$

$\Rightarrow \quad y = 2$

$\therefore \quad y - 2 + 2(x-2) = 0$

$\therefore \quad 2x + y - 6 = 0.$

7. Points are $(p, ap^2 + bp + c)$ & $(q, aq^2 + bq + c)$

$\therefore \quad \frac{y - (ap^2 + bp + c)}{x - p} = \frac{a(q^2 - p^2) + b(q - p)}{q - p}$

$\therefore \quad \dfrac{y-\left(ap^2+bp+c\right)}{x-p}=aq+ap+b$

$\therefore \quad y=\left(aq+ap+b\right)x-apq+c$

$\therefore \quad m=aq+ap+b$

& $\quad m=2ax+b=\dfrac{dy}{dx}$

$\therefore \quad x=\dfrac{p+q}{2}$

8. $\sqrt{x}+\sqrt{y}=\sqrt{a}$

$\therefore \quad \dfrac{1}{2\sqrt{x}}+\dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx}=0 \qquad x-x=\sqrt{xy}$

$\therefore \quad \dfrac{dy}{dx}=-\sqrt{\dfrac{y}{x}}$

$\therefore \quad \dfrac{Y-y}{X-x}=-\sqrt{\dfrac{y}{x}}$

$X=0 \;\Rightarrow\; Y=y+\sqrt{xy}$

$Y=0 \;\Rightarrow\; X=x+\sqrt{xy}$

$x+y+2\sqrt{xy}=OA+OB=\left(\sqrt{x}+\sqrt{y}\right)^2=a$

9. $x^{2/3}+y^{2/3}=a^{2/3}$

$\therefore \quad \dfrac{2}{3x^{1/3}}+\dfrac{2}{3y^{1/3}}\dfrac{dy}{dx}=0$

$\therefore \quad \dfrac{dy}{dx}=-\left(\dfrac{y}{x}\right)^{1/3}$

$\therefore \quad \dfrac{Y-y}{X-x}=-\left(\dfrac{y}{x}\right)^{1/3}$

$\therefore \quad Y=y+x^{2/3}y^{1/3}$ When $X=0$

$X=x+x^{1/3}+y^{2/3}$ When $Y=0$

$Y^2+X^2=y^2+x^{4/3}y^{2/3}+2x^{2/3}y^{4/3}+x^2+2x^{4/3}y^{2/3}+x^{2/3}y^{4/3}$

$=x^2+y^2+3x^{4/3}y^{2/3}+3x^{2/3}y^{4/3}$

$=x^2+x^{4/3}y^{2/3}+y^2+x^{2/3}y^{4/3}+2\left(x^{4/3}y^{2/3}+x^{2/3}y^{4/3}\right)$

$=\left(x^{4/3}+y^{4/3}\right)a^{2/3}+\left(2x^{2/3}y^{2/3}\right)a^{2/3}$

$=a^{2/3}\left(x^{2/3}+y^{2/3}\right)$

$=a^2$

10. $xy^n = a^{n+1}$

$\therefore \quad y^n = nxy^{n-1}\dfrac{dy}{dx} = 0 \quad , \; n \neq -1$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{y}{nx}$

$\therefore \quad \dfrac{Y-y}{X-x} = -\dfrac{y}{nx}$

$X = 0 \;\; \Rightarrow \quad Y = y + \dfrac{y}{n}$

$Y = 0 \;\; \Rightarrow \quad X = x + nx$

$\therefore \quad \Delta = \dfrac{1}{2}XY = \dfrac{1}{2}xy(1+n)\left(1+\dfrac{1}{n}\right)$, Here n is a constant.

Δ is constant only when xy is constant but xy^n is constant

$\therefore \quad n = 1.$

11. $f(x) = 2x^3 - 9x^2 + 12x - 3$

$\therefore \quad f'(x) = 6x^2 - 18x + 12 > 0$

$\therefore \quad x^2 - 3x + 2 > 0$

$\therefore \quad x \in (-\infty, 1) \cup (2, \infty)$

12. $f(x) = x^3 - ax^2 + 48x + 19$

$f'(x) = 3x^2 - 2ax + 48 \geq 0 \quad \forall\, x$

$\therefore \quad (2a)^2 - 4(3)(48) \leq 0$

$\therefore \quad a^2 - 144 \leq 0$

$\therefore \quad a \in [-12, 12]$

13. $f(x) = 2x^3 - 9x^2 - 60x + 81$

$\therefore \quad f'(x) = 6x^2 - 18x - 60 < 0$

$\therefore \quad x^2 - 3x - 10 < 0$

$\therefore \quad x \in (-2, 5)$

14. $f(x) = \dfrac{x^2}{x+2}$

$\therefore \quad f'(x) = \dfrac{2x(x+2) - x^2}{x+2}$

$= \dfrac{x^2 + 4x}{x+2} < 0$

$\therefore \quad \frac{x(x+4)}{(x+2)} < 0 \qquad \therefore \quad x \in (-\infty, -4) \cup (-2, 0)$

15. $f(x) = x^2$

$\therefore \quad f'(x) = x^x(1 + \ell n\, x) = 0$

$\therefore \quad 1 + \ell n\, x = 0$

$\therefore \quad x = \frac{1}{e}$

For $x < \frac{1}{e}, f'(x) < 0$

$\therefore$ Function decreases in $\left(0, \frac{1}{e}\right)$.

16. $f(x) = \frac{\log x}{x}$

$\therefore \quad f'(x) = \frac{1 - \log x}{x^2} < 0$

$\therefore \quad \log x > 1 \quad \Rightarrow \quad x > e$

$\therefore \quad x \in (e, \infty)$

17. $f(x) = 2|x-2| + |x-3|$

For $x < 2$

$f(x) = 2(2-x) + 3 - x$

$= 7 - 3x$ is a decreasing function.

For $2 < x < 3$

$f(x) = 2(x-2) + 3 - x$

$= x - 1$ is increasing function.

For $x > 3$

$f(x) = 3x - 7$ is an increasing function.

$\therefore \quad x \in (2, \infty)$

18. $f(x) = \cos x - \sin x$

$\therefore \quad f'(x) = -\sin x - \cos x < 0 \qquad \forall\, x \in \left(0, \frac{\pi}{2}\right)$

$g(x) = \cos x + \sin x$

$g'(x) = \cos x - \sin x > 0 \qquad \forall\, x \in \left(0, \frac{\pi}{4}\right)$

$< 0 \qquad \forall\, x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$h(x) = \frac{\sin x}{x}$$

$$h'(x) = \frac{x\cos x - \sin x}{x^2} = 0 \quad \text{at} \quad x = 0,$$

$$h''(x) = \frac{x^2(-x\sin x) - 2x(x\cos x - \sin x)}{x^2} < 0 \qquad \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

$\therefore \quad h'(x) < 0 \qquad \forall\, x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \frac{x}{\sin x}$ being reciprocal of $\frac{\sin x}{x}$ is an increasing function.

19. $$f(x) = \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$$

$$f'(x) = \frac{(a\cos x - b\sin x)(c\sin x + d\cos x) - (a\sin x + b\cos x)(c\cos x - d\sin x)}{(c\sin x + d\cos x)^2}$$

$$= \frac{ad - bc}{(c\sin x + d\cos x)^2} > 0 \qquad \forall\, x \qquad \text{iff} \qquad ad - bc > 0.$$

20. $\sin x - bx + c = f(x)$

$f'(x) = \cos x - b \le 0 \quad \forall \ b \ge 1$

21. $y = 2x^3 - 3x^2 - 36x + 10 = f(x)$

$f'(x) = 6x^2 - 6x - 36 = 0$

$\therefore \quad x^2 - x - 6 = 0$

$\therefore \quad x = 3 \ \text{ or } \ x = -2$

$f(3) = 2(27) - 3(9) - 36(3) + 10$

$= -71$

$f(-2) = 2(-8) - 3(4) - 36(-2) + 10$

$= -28 + 82 = 54$

22. $f(x) = x^2 - 3x + 3$

$\therefore \quad f'(x) = 2x - 3 = 0$

$\therefore \quad x = \frac{3}{2} \qquad f''(x) = 2 > 0$

$\therefore \quad f(x)$ has minima at $x = \frac{3}{2}$

$$f\left(\frac{3}{2}\right)=\frac{9}{4}-\frac{9}{2}+3=\frac{3}{4}$$

23. $f(x)=2x^3-3x^2-12x+8$

$f'(x)=6x^2-6x-12=0$

$\therefore \quad x=2 \quad \text{or} \quad x=-1$

$f''(x)=12x-6 \quad <0 \quad \text{for} \quad x=-1$

and $\quad >0 \quad \text{for} \quad x=2$

$\therefore \quad x=2$ is minima

24. $a^2\sec^2 x+b^2\operatorname{cosec}^2 x=f(x)$

$f'(x)=2a^2\sec^2 x\tan x-2b^2\operatorname{cosec}^2 x\cot x=0$

$f(x)$ can have minima only as maxima is ∞.

$\therefore \quad a^2\sec^2 x\tan x=b^2\operatorname{cosec}^2 x\cot x$

$\therefore \quad a^2\frac{\sin x}{\cos^3 x}=b^2\frac{\cos x}{\sin^3 x}$

$\therefore \quad \tan^4 x=\frac{b^2}{a^2}$

$\therefore \quad \tan^2 x=\left|\frac{b}{a}\right|, \quad \cot^2 x=\left|\frac{a}{b}\right|$

For $a,b>0$

$\sec^2 x=\frac{a+b}{a}, \operatorname{cosec}^2 x=\frac{a+b}{b}$

$\therefore \quad a^2\sec^2 x+b^2\operatorname{cosec}^2 x$

$=a^2+ab+ab+b^2=(a+b)^2$

25. $f(x)=\sin x+\sin x\cos x$

$f'(x)=\cos x+\cos^2 x-\sin^2 x=0 \qquad =\cos x+\cos 2x$

$\therefore \quad \cos x+2\cos^2 x-1=0$

$\therefore \quad \cos x=-1 \Rightarrow x=\pi \quad \text{or} \quad \cos x=\frac{1}{2} \Rightarrow x=\frac{\pi}{3} \text{ or } -\frac{\pi}{3}$

$f''(x)=-\sin x-2\sin 2x \quad <0 \quad \text{for} \quad x=\frac{\pi}{3}$

$>0 \quad \text{for} \quad x=-\frac{\pi}{3}$

$\therefore \quad x=\frac{\pi}{3}$

26. $f(x) = x + \sin x$

$f'(x) = 1 + \cos x \qquad f''(x) = -\sin x$

When $f'(x) = 0$, $f''(x) = 0$

$\therefore$ $f(x)$ has neither minimum nor maximum.

27. $\Delta = \frac{1}{2}\begin{vmatrix} a & 0 & 1 \\ a\cos\theta & b\sin\theta & 1 \\ a\cos\theta & -b\sin\theta & 1 \end{vmatrix}$

$= \frac{1}{2}(2ab\sin\theta - 2ab\cos\theta\sin\theta)$

$= ab(\sin\theta - \sin\theta\cos\theta)$

$\therefore \quad \frac{d\Delta}{d\theta} = ab(\cos\theta - \cos 2\theta) = 0$

$\therefore \quad 2\cos^2\theta - \cos\theta - 1 = 0$

$\cos\theta = 1 \quad \text{or} \quad \cos\theta = -\frac{1}{2}$

Now for a triangle,as it will be a st.line, $\theta \neq 0$, $\therefore \quad \cos\theta \neq 1$

$\therefore \quad \cos\theta = -\frac{1}{2}$

$\therefore \quad \theta = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad \Delta = ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = \frac{3\sqrt{3}ab}{4}$

28. $f'(x) = 3x^2 - 6x + 6 \quad > 0 \qquad \text{for all } x$

$\therefore$ Neither minimum nor maximum.

29. $f(x) = (x+6)^4(8-x)^3$

Let $x + 6 = t$

$\therefore \quad 8 - x = 14 - t$

$\therefore \quad f(t) = t^4(14-t)^3 \qquad \& \ f'(x) = f'(t)$

$\therefore \quad f'(t) = 4t^3(14-t)^3 - 3t^4(14-t)^2 = 0$

$\therefore \quad t^3(14-t)^2\left[4(14-t) - 3t\right] = 0$

$\therefore \quad t^3(4-t)^2\left[56 - 7t\right] = 0$

$\therefore \quad t = 0 \ \text{or} \ t = 14 \ \text{or} \ t = 8$

Now $t \neq 0$ & $t \neq 14$ as product will become zero

$\therefore \quad t = 8$ & $f(t) = 8^4.6^3$

30. $x^2-(a-2)x-(a+1)=0$

$\alpha+\beta=a-2, \alpha\beta=-(a+1)$

$\therefore \quad \alpha^2+\beta^2=a^2-4a+4+2a+2$

$\therefore \quad f(a)=a^2-2a+6$

$\therefore \quad f'(a)=2a-2=0 \qquad f''(a)>0$

$\therefore \quad a=1$

$\therefore \quad f(a)=5=\min(\alpha^2+\beta^2)$

31. Function must be continuous and differentiable to apply Rolle's theorem.

32. Function must be continuous and differentiable to apply Rolle's theorem.

33. $f(x)=\log(\sin x) \qquad \text{in } \left[\frac{\pi}{6},\frac{5\pi}{6}\right]$

$f\left(\frac{\pi}{6}\right)=f\left(\frac{5\pi}{6}\right)$

$f'(x)=\cot x=0 \quad \text{at } x=\frac{\pi}{2}$

$\therefore \quad c=\frac{\pi}{2}$

34. $f(x)=\log\left(\frac{x^2+ab}{x(a+b)}\right) \text{ in } [a,b]$

$f'(x)=\frac{2x}{x^2+ab}-\frac{(a+b)}{x(a+b)}=0$

$\therefore \quad 2x^2=x^2+ab$

$\therefore$ x = GM of a & b.

35. $f(x)=x^3+bx^2+ax$ satisfies Rolle's theorem on $[1,3]$

$c=2+\frac{1}{\sqrt{3}}$

$\therefore \quad f(1)=f(3)$

$\therefore \quad 1+a+b=27+9b+3a$ and $3\left(1+\frac{1}{\sqrt{3}}\right)^2+2b\left(2+\frac{1}{\sqrt{3}}\right)+a=0$

Solving , we get $(a,b)=(11,-6)$

36. $f(x)=\log x \text{ in } [1,e]$

$f'(x)=\frac{1}{x} \qquad f'(c)=\frac{1}{c}$

$\therefore \quad \frac{1}{c}=\frac{\log e-\log 1}{e-1}=\frac{1}{e-1}$

$\therefore \quad c=e-1.$

37. Here, $f(0)=f(2)=0$ in $[0,2]$

$\therefore$ $f'(c)=0$

$\therefore$ $(c-2)^2+2c(c-2)=0$

$\therefore$ $(c-2)[3c-2]=0$

$\therefore$ $c=\dfrac{2}{3}$ as $c\in(0,2)$

38. $f(x)=\ell x^2+mx+n$ in

$\therefore$ $f'(c)=2\ell x+m=\dfrac{\ell(b^2-a^2)+m(b-a)}{b-a}$

$\therefore$ $2\ell x+m=\ell(b+a)+m$

$\therefore$ $x=\dfrac{a+b}{2}$

39. Function should be diffrentiable in domain.

40. $\sqrt{x+1}-\sqrt{x}=\dfrac{1}{\sqrt{x}+\sqrt{x+1}}$

Now , $x>N^2$,

$\therefore$ $x+1>N^2$

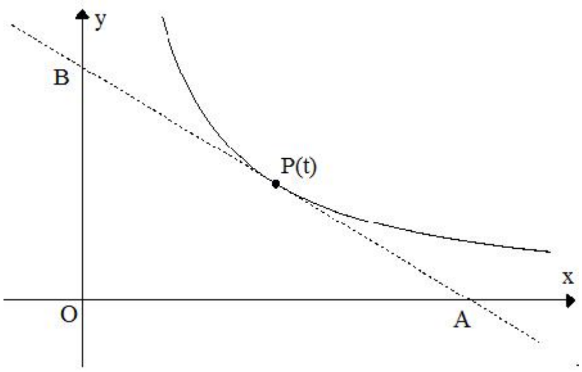
$\therefore$ $\sqrt{x}>N, \sqrt{x+1}>N$

$\therefore$ $\sqrt{x}+\sqrt{x+1}>2N$

$\therefore$ $\dfrac{1}{\sqrt{x}+\sqrt{x+1}}<\dfrac{1}{2N}$

APPLICATIONS OF DERIVATIVES
EXERCISE 1(C)

1. Let P be $\left(ct, \frac{c}{t}\right)$, then



$xy = c^2 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ or $\frac{dy}{dx} = -\frac{1}{t^2}$

tangent at P will be

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ or $x + t^2 y = 2ct$

Now $OA = |2ct|, OB = \left|\frac{2c}{t}\right|$

$\Delta = \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = 2c^2$

2. $x^2 = 4y$

$\therefore \quad \frac{dy}{dx} = \frac{x}{2}$

$\therefore \quad -\frac{dx}{dy} = -\frac{2}{x} = -2$ at $(1,2)$

FInd equation of line passing through (1,2) with slope -2.

3. $\frac{dy}{dx} = \frac{a\theta \sin\theta}{a\theta\cos\theta} = \tan\theta$

$\therefore \quad -\frac{dx}{dy} = -\cot\theta$

$\therefore \quad \frac{Y - a(\sin\theta - \theta\cos\theta)}{X - a(\cos\theta + \theta\sin\theta)} = -\cot\theta$

$\therefore \quad Y - a\sin\theta + a\theta\cos\theta = -\cot\theta X + a\cos\theta\cot\theta + a\theta\cos\theta$

$\therefore \quad Y + X\cot\theta - a(\sin\theta + \cos\theta\cot\theta) = 0$

$\therefore \quad$ Distance from origin $= \frac{|a(\sin\theta + \cos\theta\cot\theta)|}{\sqrt{1 + \cot^2\theta}}$

$= a$

4. $\frac{dy}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta$

$-\frac{dy}{dx} = -\tan\theta$

Equation of tangent is

$$\frac{y-ae^{\theta}(\sin\theta+\cos\theta)}{x-ae^{\theta}(\sin\theta-\cos\theta)}=\cot\theta$$

$\therefore \quad y-ae^{\theta}\sin\theta-ae^{\theta}\cos\theta=x\cot\theta-ae^{\theta}\cos\theta+ae^{\theta}\cos\theta\cot\theta$

$\therefore \quad \frac{x\cos\theta}{\sin\theta}-y+ae^{\theta}\sin\theta+ae^{\theta}\frac{\cos^2\theta}{\sin\theta}=0$

$\therefore \quad x\cos\theta-y\sin\theta+ae^{\theta}=0$

$\therefore \quad p=\frac{|ae^{\theta}|}{1}=ae^{\theta}$

Equation of normal is

$$\frac{y-ae^{\theta}(\sin\theta+\cos\theta)}{x-ae^{\theta}(\sin\theta-\cos\theta)}=-\tan\theta$$

$\therefore \quad y-ae^{\theta}\sin\theta-ae^{\theta}\cos\theta=-x\tan\theta+ae^{\theta}\sin\theta\tan\theta-ae^{\theta}\sin\theta$

$\therefore \quad y\cos\theta+x\sin\theta-ae^{\theta}=0$

$\therefore \quad q=\frac{|-ae^{\theta}|}{1}=ae^{\theta}$

$\therefore \quad p=q$

5. $x^{2/3}+y^{2/3}=a^{2/3}$

$\therefore \quad \frac{2}{3x^{1/3}}+\frac{2}{3y^{1/3}}\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}$

$\therefore \quad -\frac{dx}{dy}=\left(\frac{x}{y}\right)^{1/3}$

Equation of tangent is

$$\frac{Y-y}{X-x}=-\left(\frac{y}{x}\right)^{1/3}$$

$\therefore \quad x^{1/3}Y-x^{1/3}y=-y^{1/3}X+xy^{1/3}$

$\therefore \quad y^{1/3}X+x^{1/3}Y-x^{1/3}y-xy^{1/3}=0$

$$p=\frac{|x^{1/3}y+xy^{1/3}|}{\sqrt{x^{2/3}+y^{2/3}}}$$

$$=|x^{1/3}y^{1/3}a^{1/3}|$$

Equation of normal is

$$\frac{Y-y}{X-x}=\left(\frac{x}{y}\right)^{1/3}$$

$\therefore \quad y^{1/3}Y-y^{4/3}=x^{1/3}X-x^{4/3}$

$\therefore \quad x^{1/3}X-y^{1/3}Y-x^{4/3}+y^{4/3}=0$

$\therefore \quad q=\frac{\left|y^{4/3}-x^{4/3}\right|}{\sqrt{x^{2/3}+y^{2/3}}}$

$\quad =\left|\left(x^{2/3}-y^{2/3}\right)a^{1/3}\right|$

$\therefore \quad 4p^2+q^2=4x^{2/3}y^{2/3}+a^{2/3}\left(x^{4/3}-2x^{2/3}y^{2/3}+y^{4/3}\right)$

$\quad =a^{2/3}\left(x^{2/3}+y^{2/3}\right)$

$\quad =a^2.$

6. $y^2=2x, x^2+y^2=8$

$\therefore \quad x^2+2x-8=0$ and $x>0$ as $x=\frac{y^2}{2}$

$\therefore \quad x=2$

$\Rightarrow \quad y=2$

For $y^2=2x$

$\therefore \quad 2y\frac{dy}{dx}=2$

$\therefore \quad m_1=\frac{1}{y}=\frac{1}{2}$

For $x^2+y^2=8$

$\therefore \quad 2x+2y\frac{dy}{dx}=0$

$\therefore \quad m_2=-\frac{x}{y}=-1$

$\therefore \quad \tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=3$

7. $y=\frac{x+3}{x^2+1}$

$\therefore \quad \frac{dy}{dx}=\frac{\left(x^2+1\right)-\left(2x^2+6x\right)}{\left(x^2+1\right)^2}=\frac{-x^2-6x+1}{\left(x^2+1\right)^2}$

$\therefore \quad m_1 = \frac{-4-12+1}{25} = \frac{-3}{5}$

$y = \frac{x^2-7x+11}{x-1}$

$\therefore \quad \frac{dy}{dx} = \frac{\left(2x^2-9x+7\right)-\left(x^2-7x+11\right)}{\left(x-1\right)^2} = \frac{x^2-2x-4}{\left(x-1\right)^2}$

$\therefore \quad m_2 = \frac{4-4-4}{1} = -4$

$\therefore \quad \tan\alpha = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = 4$

8. $x^3-3xy^2+2=0$

$\therefore \quad 3x^2-3xy^2-6xy\frac{dy}{dx}=0$

$\therefore \quad x^2-y^2 = 2xy\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{x^2-y^2}{2xy} = m_1$

$3x^2y-y^3+2=0$

$\therefore \quad 6xy+3x^2\frac{dy}{dx}-3y^2\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}\left(x^2-y^2\right) = -2xy$

$\therefore \quad \frac{dy}{dx} = \frac{-2xy}{x^2-y^2} = m_2$

$m_1m_2 = -1$

9. $x = y^2$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2y} = m_1$

$xy = k$

$\therefore \quad \frac{dy}{dx} = -\frac{y}{x} = m_2$

$m_1m_2 = -1$

$\therefore \quad \frac{-1}{2x} = -1$

$\therefore \quad x = \frac{1}{2}$

$\therefore \quad y = \pm\frac{1}{\sqrt{2}}$

$\therefore \quad k = \pm\frac{1}{2\sqrt{2}}$

10. $ST = \frac{3}{8}, SN = 24$

$y_0^2 = ST.SN$

$= \frac{3}{8} \times 24 = 9$

$\therefore \quad y_0 = \pm 3$

11. $by^2 = (x+a)^3$

$\therefore \quad 2by\frac{dy}{dx} = 3(x+a)^2$

$\therefore \quad \frac{dy}{dx} = \frac{3(x+a)^2}{2by} = \tan\theta$

$\cot\theta = \frac{2by}{3(x+a)^2}$

$ST = |y\cot\theta| = \left|\frac{2by^2}{3(x+a)^2}\right|$

$SN = |y\tan\theta| = \left|\frac{3(x+a)^2}{2b}\right|$

$\therefore \quad \frac{3p(x+a)^2}{2b} = \frac{4qb^2y^4}{9(x+a)^4}$

$\therefore \quad \frac{p}{q} = \frac{8b^3y^4}{27(x+a)^6} = \frac{8b}{27}\frac{(by^2)^2}{(x+a)^6} = \frac{8b}{27}$

12. $xy^n = a^{n+1}$

$\therefore \quad \frac{dy}{dx} = \frac{-y}{nx} = \tan\theta$

$\therefore \quad SN = |y\tan\theta|$

$= \left|\frac{-y^2}{nx}\right| = \text{constant}$

But xy^n = constant

$\Rightarrow \quad n = -2$

13. $x^2y^2 = a^5$

$\therefore \quad 2xy^2 + 2x^2y\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{-y}{x} = \tan\theta$

$\therefore \quad \cot\theta = \frac{-x}{y}$

$\therefore \quad ST = |y\cot\theta| = |-x|$

14. $x^m y^n = a^{m+n}$

$\therefore \quad mx^{m-1}y^n + nx^m y^{n-1}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{my}{nx}$

$\therefore \quad \cot\theta = -\frac{nx}{my}$

$\therefore \quad ST = \left|-\frac{nx}{m}\right|$

15. From informationin Q.22,

$(ST)^2 \propto (SN)$

16. $-\frac{dx}{dy} = -\frac{a\theta\cos\theta}{a\theta\sin\theta} = -\cot\theta$

As shown in Q.14 of this exercise it is at a constant distance from origin.

17. $ax + by + c = 0$ normal to $xy = 1$

For $xy = 1, \frac{dy}{dx} = -\frac{y}{x}$

As xy is positive, $\frac{dy}{dx} < 0 \quad \forall\, x, y$

$\therefore \quad -\frac{dx}{dy} < 0 \quad \forall\, x, y$

$\therefore$ Slope of normal is positive.

$\therefore \quad a > 0, b < 0$ or $a < 0, b > 0$

18. $(3-a)x + ay + a^2 - 1 = 0$

$\therefore \quad -\left(\frac{a}{3-a}\right) > 0$

$\therefore \quad a \in (-\infty, 0) \cup (3, \infty)$

19. $f(x) = 2x^2 - \log|x|$

$\therefore \quad f'(x) = 4x - \frac{1}{x} < 0$

$\therefore \quad \frac{4x^2 - 1}{x} < 0$

$\therefore \quad \frac{(2x+1)(2x-1)}{x} < 0 \qquad \therefore \quad x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

20. $f(x) = \frac{x}{\sin x}$, $g(x) = \frac{x}{\tan x}$

$f'(x) = \frac{\sin x - x\cos x}{x^2} > 0 \quad \forall\, x \in (0,1)$

$g'(x) = \frac{\tan x - x\sec^2 x}{x^2} < 0 \quad \forall\, x \in (0,1)$

21. $f(x) = \tan^{-1}(\sin x + \cos x)$

Let $\quad g(x) = \tan^{-1} x$

$\therefore \quad g'(x) = \frac{1}{1+x^2} > 0 \quad \forall\, x$

$\therefore \quad f(x)$ increases when $\sin x + \cos x$ increases

Let $\quad h(x) = \sin x + \cos x$

$\therefore \quad h'(x) = \cos x - \sin x > 0$

$\therefore \quad \cos x > \sin x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

22. $f(x) = x^{100} + \sin x - 1$

$f'(x) = 100x^{99} + \cos x < 0$

23. $f(x) = |x| - |x-1|$

$x < 0$	$\Rightarrow$	$f(x) = -x + 1 - x = 1 - 2x$	MD
$0 < x < 1$	$\Rightarrow$	$f(x) = x + 1 - x = 1$	Constant
$x > 1$	$\Rightarrow$	$f(x) = 2x - 1$	MI

24. $f(x) = x(a^2 - 2a - 2) + \cos x$

$f'(x) = a^2 - 2a - 2 - \sin x > 0 \quad \forall\, x$

$\therefore \quad a^2 - 2a - 2 > 1$

$\therefore \quad a \in (-\infty, -1) \cup (3, \infty)$

25. $\phi(x) = 3f\left(\frac{x^3}{3}\right) + f(3 - x^2) \;\; \forall \;\; x \in (-3, 4)$

$f''(x) > 0$

$\therefore \quad \phi'(x) = 3x^2 f'\left(\frac{x^3}{3}\right) - 2x\, f'(3 - x^2) > 0$

$x^2 f'\left(\frac{x^3}{3}\right) > 2x\; f'(3 - x^2)$

For $x > 0$

$x\, f'\left(\frac{x^3}{3}\right) > 2\, f'(3 - x^2)$

$\therefore \quad \frac{x}{2} f'\left(\frac{x^3}{3}\right) > f'(3 - x^2)$

26. $f'(x) \ge 0$, $g'(x) \le 0$

$\therefore \quad h'(x) = f'(\underset{>0}{g(x)})\, \underset{<0}{g'(x)} \le 0$

$\therefore \quad h(2) = 1$ as $h(1) = 1$

27. $y = a \log|x| + bx^2 + x$

$\therefore \quad \frac{dy}{dx} = \frac{a}{x} + 2bx + 1 = 0$

$\therefore \quad \frac{a + 2bx^2 + x}{x} = 0 \qquad\qquad \alpha = \frac{-4}{3}, \beta = 2$

$\therefore \quad \alpha + \beta = \frac{2}{3} = \frac{-1}{2b}$

$\therefore \quad b = \frac{-3}{4}$

$\alpha\beta = \frac{-8}{3} = \frac{a}{2b}$

$\therefore \quad a = \frac{-8}{3} \times 2 \times \frac{-3}{4} = 4$

28. Point on $y^2 = 4x$ is $(t^2, 2t)$

Distance between point & (2,1) is

$d = \sqrt{(t^2-2)^2 + (2t-1)^2}$

$d^2 = (t^2-2)^2 + (2t-1)^2$

$= t^4 - 4t + 5 \qquad = f(t)$

$\therefore \quad f'(t) = 4t^3 - 4 = 0$

$\therefore \quad t = 1$ we can show that $t = 1$ is minima

$\therefore \quad$ Point is $(1,2)$.

28. Point nearest to the required line will have common normal.

$\therefore \quad \frac{dy}{dx} = 3 = 2x + 7$

$\therefore \quad x = -2, y = -8$

point is $(-2,-8)$

30. $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$

$x = a\cos\theta, y = 2\sin\theta$

$\sqrt{a^2\cos^2\theta + 4(1-\sin\theta)^2} = d$

$d^2 = f(\theta) = a^2\cos^2\theta + 4 - 8\sin\theta + 4\sin^2\theta$

$= a^2 + 4 + (4-a^2)\sin^2\theta - 8\sin\theta$

$\therefore \quad f'(\theta) = 2(4-a^2)\sin\theta\cos\theta - 8\cos\theta = 0$

$\therefore \quad \cos\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{2}$

$\therefore \quad$ point is (0,2).

31. $r\theta + 2r = k$

$\therefore \quad r(\theta + 2) = k$

$\therefore \quad \theta = \frac{k-2r}{r}$

$\therefore \quad A = \frac{1}{2} r^2 \times \frac{(k-2r)}{r}$

$= \frac{kr - r^2}{2}$

$\therefore \quad \frac{dA}{dr} = 0$

$\therefore \quad r = \frac{k}{4}$

$\therefore \quad \theta = \frac{k - \frac{k}{2}}{\frac{k}{4}} = 2^c$

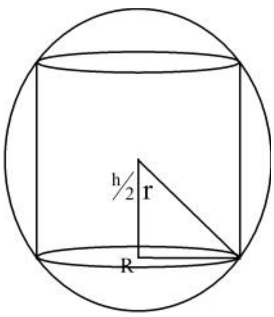
32. From above example, $\theta = 2^c$

$\therefore \quad 2r + 2r = 20$

$\therefore \quad r = 5$

$\therefore \quad A = \frac{1}{2} \times 25 \times 2 = 25$ sq.cm.

33.



$R^2 + \frac{h^2}{4} = r^2$

$v = \pi R^2 h$

$= \pi h \left(r^2 - \frac{h^2}{4} \right)$

$= \pi r^2 h - \frac{\pi h^3}{4}$

$\therefore \quad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4} = 0$

$\therefore \quad h^2 = \frac{4r^2}{3} \qquad h = \frac{2r}{\sqrt{3}}$

34. $s = 2\pi r(r+h)$

$= 2\pi r\left(r + \frac{v}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{2v}{r}$

$\frac{ds}{dr} = 0 \qquad \therefore \qquad 4\pi r - \frac{2v}{r^2} = 0$

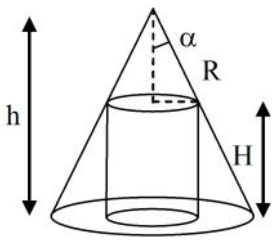
$\therefore \qquad r^3 = \frac{v}{2\pi} \qquad \therefore \qquad r = \left(\frac{v}{2\pi}\right)^{\frac{1}{3}}$

$\pi r^2 = \left(\frac{\pi v^2}{4}\right)^{\frac{1}{3}}$

$\frac{v}{\pi r^2} = \left(\frac{4v}{\pi}\right)^{\frac{1}{3}} = h$

$h = 2r$

35. $\frac{R}{h-H} = \tan\alpha$



$R = \tan\alpha(h-H)$

curved surface area

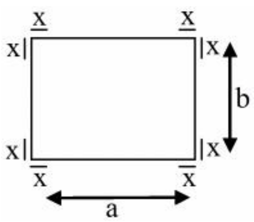
$= S_c = 2\pi RH$

$= 2\pi\tan\alpha\left(hH - H^2\right)$

$\therefore \qquad \frac{dS_c}{dH} = 2\pi\tan\alpha(h-2H) = 0$

$\therefore \qquad H = \frac{h}{2}\cdot$

36.



$V = (a-2x)(b-2x)x$

$V = 4x^3 - 2(a+b)x^2 + abx$

$\therefore \qquad \frac{dV}{dx} = 12x^2 - 4(a+b)x + ab = 0$

$$\therefore \quad x = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$$

$$= \frac{(a+b) \pm \sqrt{a^2 + b^2 - ab}}{6}$$

But $\quad x < a\,,\, x < b$

$$\therefore \quad x = \frac{(a+b) - (a^2 + b^2 - ab)^{\frac{1}{2}}}{6} = \frac{1}{6}\left[(a+b) - (a^2 + b^2 - ab)^{\frac{1}{2}}\right]$$

37. $v = x(a - 2x)^2$

$$\therefore \quad v = 4x^3 - 4ax^2 + a^2x$$

$$\therefore \quad \frac{dv}{dx} = 12x^2 - 8ax + a^2 = 0 \qquad \therefore \quad x = \frac{a}{2} \quad \text{or} \quad x = \frac{a}{6}$$

But $x = \frac{a}{2}$ will make volume zero.

$$\therefore \quad x = \frac{a}{6}$$

38. $a^2h = 32$

$a^2 + 4ah$ has to be minimised

$$\therefore \quad h = \frac{32}{a^2}$$

$$\therefore \quad f(a) = a^2 + \frac{128}{a}$$

$$\therefore \quad f'(a) = 2a - \frac{128}{a^2} = 0$$

$$\therefore \quad a = 4 \;\; \& \;\; h = 2$$

$$\therefore \quad \text{Area} = 16 + 32 = 48$$

39. Line is $(y - 4) = m(x - 3)$

$$x = 0 \quad \Rightarrow \quad y = 4 - 3m$$

$$y = 0 \quad \Rightarrow \quad x = 3 - \frac{4}{m}$$

$$\Delta = \frac{1}{2}(4 - 3m)\left(3 - \frac{4}{m}\right)$$

$$= \frac{1}{2}\left(24 - 9m + \frac{16}{m}\right)$$

$$\therefore \quad \frac{d\Delta}{dm} = \frac{-9}{2} + \frac{16}{2m^2} = 0$$

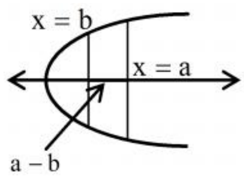
$\therefore \quad \frac{8}{m^2} = \frac{9}{2}$

$\therefore \quad m^2 = \frac{16}{9}$

$\therefore \quad m = \frac{-4}{3}$ as $m > 0 \quad \Rightarrow$ no Δ is formed

$\therefore \quad \Delta = \frac{1}{2}(8)(6) = 24$

40.



$x = b \quad \Rightarrow \quad y^2 = 4ab \qquad y = \pm\sqrt{4ab} = \pm 2\sqrt{ab}$

$\therefore \quad |2y| = \pm 4\sqrt{ab}$

$A = \frac{1}{2}(a-b)\left(4a + 4^2\sqrt{ab}\right)$

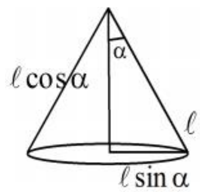
$= 2a^2 + 2a^{\frac{3}{2}}b^{\frac{1}{2}} - 2ab - 2a^{\frac{1}{2}}b^{\frac{3}{2}}$

$\frac{dA}{sb} = \frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}} - 2a - 3a^{\frac{1}{2}}b^{\frac{1}{2}} = 0$

$\therefore \quad -3a^{\frac{1}{2}}b - 2ab^{\frac{1}{2}} + a^{\frac{3}{2}} = 0$

$\Rightarrow \quad b = \frac{a}{9}$

41. $\therefore \quad V = \frac{1}{3}\pi \ell^3 \sin^2\alpha\cos\alpha$



$\frac{dV}{d\alpha} = \frac{1}{3}\pi\ell^2\left[2\sin\alpha\cos^2\alpha - \sin^3\alpha\right] = 0$

$\therefore \quad \sin\alpha = 0$ or $2\cos^2\alpha = \sin^2\alpha$
Rejected

$\therefore \quad \tan\alpha = \sqrt{2}$ as $\alpha \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \alpha = \tan^{-1}\left(\sqrt{2}\right)$

42. $V = \frac{1}{3}\pi r^2 h = \text{constant}$

$S_c = \pi r\sqrt{r^2 + h^2}$ has to be maximized

$\therefore \quad h = \dfrac{3V}{\pi r^2}$

$\therefore \quad S_c = \pi r^2\sqrt{r^2 + \dfrac{9V^2}{\pi^2 r^4}}$

$= \sqrt{\pi^2 r^4 + \dfrac{9V^2}{r^2}}$

$\dfrac{dS_c}{dr} = \dfrac{dS_c^2}{dr} = 0$

$\therefore \quad 4\pi^2 r^3 - \dfrac{18V^2}{r^3} = 0$

$\therefore \quad 4\pi^2 r^6 = 18V^2$

$\therefore \quad r^6 = \dfrac{9V^2}{2\pi^2}$

$\therefore \quad r = \dfrac{3^{\frac{1}{3}}}{2^{\frac{1}{6}}} \dfrac{V^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}$

$r^2 = \left(\dfrac{9V^2}{2\pi^2}\right)^{\frac{1}{3}}$

$h = \dfrac{3V}{\pi r^2} = \dfrac{3V}{\pi} \times \left(\dfrac{2\pi^2}{9V^2}\right)^{\frac{1}{3}}$

$= \dfrac{3V}{\pi} \times \dfrac{2^{\frac{1}{3}}\pi^{\frac{2}{3}}}{9^{\frac{1}{3}}V^{\frac{2}{3}}} = \dfrac{\left(3^{1/3}\right)\left(2^{1/3}\right)V^{1/3}}{\pi^{1/3}}$

$\dfrac{h}{r} = 2^{1/3} \times 2^{1/6} = \sqrt{2}$

43. $\ell^2 = h^2 + r^2$

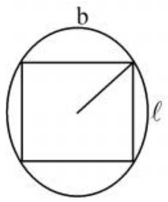
$r = \sqrt{\ell^2 - h^2}$

$V = \dfrac{1}{3}r^2 h = \dfrac{1}{3}\left(\ell^2 - h^2\right)h$

$= \dfrac{\ell^2 h}{3} - \dfrac{h^3}{3}$

$\therefore \quad \dfrac{dV}{dh} = \dfrac{\ell^2}{3} - h^2 = 0$

$\therefore \quad h = \dfrac{\ell}{\sqrt{3}}$

44. 

$b^2 + \ell^2 = 4r^2$

$b = \sqrt{4r^2 - \ell^2}$

$S = kb\ell^3$

$\quad = k\ell^3 \sqrt{4r^2 - \ell^2}$

$\therefore \quad \dfrac{dS}{d\ell} = 3k\ell^2\sqrt{4r^2 - \ell^2} - \dfrac{k\ell^4}{\sqrt{4r^2 - \ell^2}} = 0$

$\therefore \quad 3k\ell^2\left(4r^2 - \ell^2\right) = k\ell^4$

$\therefore \quad \ell = 0$ Rejected or

$\quad 3\left(4r^2 - \ell^2\right) = \ell^2$

$\therefore \quad 12r^2 = 4\ell^2$

$\therefore \quad \ell = \sqrt{3}r$

$\therefore \quad b = r$

45. $b^2 + d^2 = 4r^2$

$d^2 = 4r^2 - b^2$

$\therefore \quad S = k\,b\,d^2$

$\quad = kb\left(4r^2 - b^2\right) = 4kbr^2 - kb^3$

$\therefore \quad \dfrac{dS}{dr} = 4kr^2 - 3kb^2 = 0$

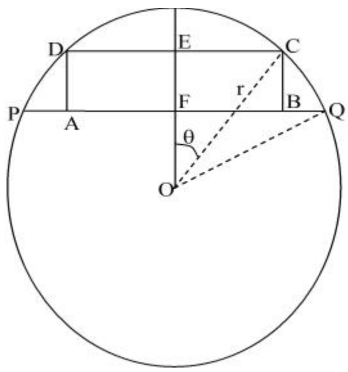
$\therefore \quad b = \dfrac{2r}{\sqrt{3}}$

$\therefore \quad d^2 = 4r^2 - \dfrac{4r^2}{3} = \dfrac{8r^2}{3}$

$\Rightarrow \quad d = \dfrac{2\sqrt{2}\,r}{3}$

$\therefore \quad d = \sqrt{2}\,b = 2\sqrt{\dfrac{2}{3}}r$

46. Let OABC be the sheet of paper as



The corner A of the rectangular sheet OABC is folded over along PQ so as to reach the opposite edge OC at R.
Let the crease PQ be of length x.
Let $\angle APQ = \theta$. Then $\angle PQR = \theta$ and $\angle OPR = \pi - 2\theta$.
In ΔAPQ , we have

$$\cos\theta = \frac{AP}{PQ}$$

$\Rightarrow \quad AP = x\cos\theta$
In ΔOPR, we have

$$\cos(\pi - 2\theta) = \frac{OP}{RP}$$

$\Rightarrow \quad -\cos 2\theta = \frac{OP}{AP} \qquad [\because\ AP = RP]$

$\Rightarrow \quad OP = -AP\cos 2\theta = -x\cos\theta\cos 2\theta$
Now,

$a = OA = OP + AP$

$\Rightarrow \quad a = x\cos\theta - x\cos 2\theta\cos\theta$

$\Rightarrow \quad x = \frac{a}{\cos\theta - \cos\theta\cos 2\theta} \qquad ...(i)$

$\Rightarrow \quad \frac{a}{x} = \cos\theta - \cos\theta\cos 2\theta$

Let $y = \frac{a}{x}\cdot$ Then y is maximum when x is minimum.
Now,
$y = \cos\theta - \cos\theta\cos 2\theta$

$\Rightarrow \quad \frac{dy}{d\theta} = -\sin\theta + \sin\theta\cos 2\theta + 2\cos\theta\sin 2\theta$

For maximum or minimum values of y we must have $\frac{dy}{d\theta} = 0$

$\Rightarrow \quad -\sin\theta + \sin\theta\cos 2\theta + 4\sin\theta\cos^2\theta =$

$\Rightarrow \quad -\sin\theta(1 - \cos 2\theta) + 4\sin\theta(1 - \sin^2\theta) = 0$

$\Rightarrow \quad -2\sin^3\theta + 4\sin\theta - 4\sin^3\theta = 0$

$\Rightarrow \quad 4\sin\theta = 6\sin^3\theta$

$\Rightarrow \quad \sin^2\theta = \frac{2}{3}$ or $\sin\theta = 0$

$\Rightarrow \quad \sin\theta = \sqrt{\frac{2}{3}}$ or $\theta = 0.$

Now,

$$\frac{d^2y}{d\theta^2} = -\cos\theta + \cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta + 4\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta$$

$$\Rightarrow \quad \frac{d^2y}{d\theta^2} = -\cos\theta + 5\cos\theta\cos 2\theta - 4\sin\theta\sin 2\theta$$

For $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \sqrt{\frac{2}{3}}$, we have

$$\frac{d^2y}{d\theta^2} = -\frac{1}{\sqrt{3}} + 5\times\sqrt{\frac{2}{3}}\times\left(\frac{2}{3}-1\right) - 4\times\sqrt{\frac{2}{3}}\times 2\sqrt{\frac{2}{3}}\times\sqrt{\frac{1}{3}} < 0.$$

So, y is maximum when $\sin\theta = \sqrt{\frac{2}{3}}$

Hence, x is minimum when $\sin\theta = \sqrt{\frac{2}{3}}$

Putting $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \sqrt{\frac{1}{3}}$ in (i), we get

$$\text{Length of the crease} = x = \frac{a}{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\left(1 - 2\times\frac{2}{3}\right)} = \frac{3\sqrt{3}a}{4}$$

47. Let speed of boat be v & walking speed be v sec α

$$\therefore \quad t = \frac{\sqrt{a^2 + (b-x)^2}}{v} + \frac{x\cos\alpha}{v}$$

$$= \frac{\sqrt{a^2 + (b-x)^2} + x\cos\alpha}{v}$$

$$\therefore \quad v\frac{dt}{dx} = \cos\alpha + \frac{1}{2\sqrt{a^2 + (b-x)^2}} \times -2(b-x) = 0$$

$$\therefore \quad \cos\alpha = \frac{b-x}{\sqrt{a^2 + (b-x)^2}}$$

$$\therefore \quad (b-x)^2 = \cos^2\alpha\left(a^2 + b^2 - 2bx + x^2\right)$$

$\therefore \quad (b-x)^2 = a^2\cot^2\alpha$

$\therefore \quad x = b - a\cot\alpha = \dfrac{b\sin\alpha - a\cos\alpha}{\sin\alpha}$

48. $\therefore \quad T = \dfrac{\sqrt{d^2+x^2}}{u} + \dfrac{1-x}{v}$

$\therefore \quad \dfrac{dT}{dx} = \dfrac{x}{u\sqrt{d^2+x^2}} - \dfrac{1}{v} = 0$

$\therefore \quad xv = u\sqrt{d^2+x^2}$

$\therefore \quad x^2\left(v^2-u^2\right) = u^2d^2$

$\therefore \quad x = \dfrac{ud}{\sqrt{v^2-u^2}}$

For students. [Think for solution if $u > v$]

49. $2\ell + 2\pi r = 440$

$\therefore \quad \ell + \pi r = 220$ & $\ell = 220 - \pi r$

$A = 2\left(220r - \pi r^2\right) \qquad = 2\ell r$

$\dfrac{dA}{dr} = 2\left(220 - 2\pi r\right) = 0$

$\therefore \quad r = 35$ ft

$\Rightarrow \quad 2r = 70$ ft & $\ell = 110$ ft

50. $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots\ldots + a_nx^{2n}$

$0 < a_1 < a_2 < \ldots\ldots < a_n$

$\therefore \quad P'(x) = 2na_nx^{2n-1} + \ldots. + 4a_2x^3 + 2a_1x$

$= 0$ only at $x = 0$ &

$P''(x) > 0 \quad \forall \; x \in R$

$\therefore \quad P(x)$ has only one minimum.

51. $x = a\sec\theta, y = b\,\text{cosec}\,\theta$

Minimum radius vector = ?

$r = \sqrt{x^2+y^2} = \sqrt{a^2\sec^2\theta + b^2\,\text{cosec}^2\,\theta}$

From (Q.4),

Minimum value of $r = \sqrt{(a+b)^2} = a + b$

52. From (Q.18)

$s = 2\pi r(r+h)$

$$= 2\pi r\left(r + \frac{v}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2v}{r}$$

$\frac{ds}{dr} = 0$ $\therefore$ $4\pi r - \frac{2v}{r^2} = 0$

$\therefore$ $r^3 = \frac{v}{2\pi}$ $\therefore$ $r = \left(\frac{v}{2\pi}\right)^{\frac{1}{3}}$

$$\pi r^2 = \left(\frac{\pi v^2}{4/3}\right)^{\frac{1}{3}}$$

$$\frac{v}{\pi r^2} = \left(\frac{4v}{\pi}\right) = h$$

$h = 2r$, form this statement $h : r = 2 : 1$

53. $f(x) = (x-1)^p (x-2)^q$

$\therefore$ $f'(x) = p(x-1)^{p-1}(x-2)^q + q(x-1)^p (x-2)^{q-1}$

$f''(x) = p(p-1)(x-1)^{p-2}(x-2)^q + 2pq(x-1)^{p-1}(x-2)^{q-1} + q(q-1)(x-1)^p (x-2)^{q-2}$

If we go on taking derivatives, we find that the condition given in the question holds when (even)th derivative is non-zero for it, p & q should be even.

54. $f(x) = xe^x$

$f'(x) = xe^x + e^x = 0$

$\therefore$ $x = -1$

$f''(x) = xe^x + 2e^x > 0$ for $x = -1$

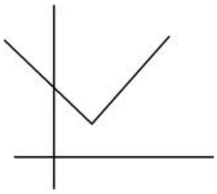
$\therefore$ $x = -1$ is a minimum

55. Time required $= T = \left(\frac{N}{x}\right)(\alpha + \beta x^2)$

$\therefore$ $T = N\left(\frac{\alpha}{x} + \beta x\right)$

$\therefore$ $\frac{dT}{dx} = N\left(\beta - \frac{\alpha}{x^2}\right) = 0$ $\therefore$ $x = \sqrt{\frac{\alpha}{\beta}}$

56. $f(x) = \max\{x, x+1, 2-x\}$

By graph, $f(x) =$ 

$$f(x) = 2 - x, x \le +\frac{1}{2}$$

$$x + 1, x > \frac{1}{2}$$

$\therefore$ $x = \frac{1}{2}$ is point of minima and minimum value is $\frac{3}{2}$.

57. $f(\alpha) = \left(1 + \frac{1}{\sin^n x}\right)\left(1 + \frac{1}{\cos^n \alpha}\right)$

$$= (1 + \sec^n \alpha)(1 + \operatorname{cosec}^n \alpha)$$

$$= 1 + \sec^n \alpha + \operatorname{cosec}^n \alpha + \sec^n \alpha \operatorname{cosec}^n \alpha$$

$\therefore$ $f'(\alpha) = n \sec^n \alpha \tan \alpha - n \operatorname{cosec}^n \alpha \cot \alpha$

$$+n \sec^n \alpha \operatorname{cosec}^n \alpha (\tan \alpha - \cot \alpha) = 0$$

$\therefore$ $\sec^n \alpha \tan \alpha (1 + \operatorname{cosec}^n \alpha) = \operatorname{cosec}^n \alpha \cot \alpha (1 + \sec^n \alpha)$

$\therefore$ $\frac{(\sec^n \alpha)(\sec^2 \alpha - 1)}{1 + \sec^n \alpha} = \frac{\operatorname{cosec}^n \alpha}{1 + \operatorname{cosec}^n \alpha}$

$\therefore$ $\frac{(\cos^n \alpha)(\sin^2 \alpha)}{(\cos^n \alpha + 1)(\cos^2 \alpha)} = \frac{\sin^n \alpha}{1 + \sin^n \alpha}$

$\therefore$ $\frac{\sin^{n-2} \alpha}{1 + \sin^n \alpha} = \frac{\cos^{n-2} \alpha}{1 + \cos^n \alpha}$

$\Rightarrow$ $\sin \alpha = \cos \alpha$

For minima,

$$\sin \alpha = \cos \alpha = \frac{+1}{\sqrt{2}}$$

$\therefore$ Minimum value $= \left(1 + 2^{n/2}\right)^2$

58. $f(x) + f\left(\frac{1}{x}\right) = \frac{1}{x}$

$\therefore$ $f(x) + f\left(\frac{1}{x}\right) = x$

$\Rightarrow$ $x = \frac{1}{x}$

$\Rightarrow \quad x = \pm 1$ Only

Here, $x = -1 \quad \Rightarrow \quad f(x) = -\frac{1}{2}$ and

$x = +1 \quad \Rightarrow \quad f(x) = +\frac{1}{2}$

$\therefore \quad f(x)$ has maximum value $\frac{1}{2}$.

59. $f(x) = \cos 2\pi x + \{x\}$

At non-integral points,

$f'(x) = -2\pi \sin 2\pi x + 1$

It tends to achieving maximum values at points infinitesimally close to and less than integers but it has a discontinuity.

$\therefore$ It has no maxima.

$f(x) = x - x^2$

$x_1 \,\&\, x_2 \in \quad y = x - x^2 \quad$ in $(0,1)$

maximum value of expression

$= \max\left(x - x^2\right) = \frac{1}{4}$

61. $f(x) = x^2, \qquad x \in [-2,-1] \cup [1,2]$

$2 - x^2, \qquad x \in (-1,1)$

$\therefore$ Function has maximum at $x = 0$ & local as well as global minima at $x = \pm 1$

62. $x^3 - ax^2 + bx - 6 = 0$ has roots real and positive

$\therefore \quad \alpha\beta\gamma = 6, \alpha + \beta + \gamma = a, \alpha\beta + \beta\gamma + \gamma\alpha = b$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$

$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{b}{6}$

Now, sum is minimum when each of them is equal

$$\frac{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}{3} \geq \left(\frac{1}{\alpha\beta\gamma}\right)^{\frac{1}{3}} \qquad \text{[AM-GM inequality]}$$

$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq \frac{3}{6^{1/3}} \qquad \therefore \quad b \geq \frac{3 \times 6}{6^{1/3}} = 3(36)^{1/3}$

63. $f'(x) = \dfrac{2}{3(6 - x)^{\frac{1}{3}}}$

Which is not diff. at $x = 6$

$\therefore$ Theorems are not applicable.

64. By definition.

65. $f(0) = -6$, $f(4) = +6$

$\therefore \quad f'(x) = (x-2)(x-3) + (x-1)(x-2) + (x-1)(x-3)$

$f'(c) = \frac{6+6}{4-0} = 3$

$\therefore \quad 3x^2 - 12x + 11 = 3$ and $x = c$

$\therefore \quad 3c^2 - 12c + 8 = 0$

$c = 12 \pm \frac{\sqrt{144-96}}{6}$

$= 12 \pm \frac{\sqrt{48}}{6} = 6 \pm \frac{2\sqrt{3}}{3} = 2 \pm \frac{2}{\sqrt{3}}$

66. $f(x) = x^{\alpha} \log x$

$f'(x) = x^{\alpha-1}(1 + \alpha \log x) = 0$

$c = e^{-1/\alpha} \in (0,1)$

$\therefore \quad \alpha > 0$

67. $a + b + c = 0$

$f(x) = ax^3 + bx^2 + cx$

Has roots 0 & 1

$\therefore \quad 3ax^2 + 2bx + c = 0$

has at least one root in $(0,1)$.

68. $f'(c) = \frac{13-5}{2} = 4$

69. Refer (Q.28) (above)

$a + b + c = 0$

$f(x) = ax^3 + bx^2 + cx$

Has roots 0 & 1

$\therefore \quad 3ax^2 + 2bx + c = 0$

70. $x^3 - 3x + a = 0$ has two roots in $[0,1]$

$f'(x) = 3x^2 - 3 \neq 0$ in $(0,1)$

$\therefore$ There is no value of a satisfying the conditions.

APPLICATIONS OF DERIVATIVES
EXERCISE 2(A)

1. $y = x^{1/3}(x-1)$

$$\frac{dy}{dx} = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot\frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}}[4x-1]$$

hence f is ↑ for $x > \frac{1}{4}$

and f ↓ for $x < \frac{1}{4}$ | $x^{2/3}$ *is always positive and at* $x = 1/4$ *the curves has a local* minima

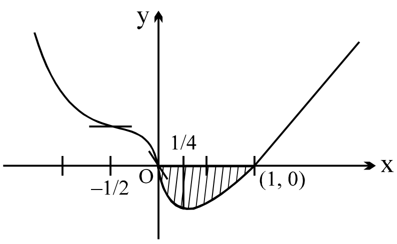
now $f'(x) = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot x^{-2/3}$ (non existent at x = 0, vertical tangent)

$$f''(x) = \frac{4}{9}\cdot\frac{1}{x^{2/3}} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{x^{5/3}}$$

$$= \frac{2}{9x^{2/3}}\left[2+\frac{1}{x}\right] = \frac{2}{9x^{2/3}}\left[\frac{2x+1}{x}\right]$$

∴ $f''(x) = 0$ at $x = -\frac{1}{2}$ (inflection point)

graph of f (x) is as



$$A = \int_0^1 \left(x^{4/3} - x^{1/3}\right)dx = \left.\frac{3}{7}x^{3/7} - \frac{3}{4}x^{4/3}\right]_0^1$$

$$= \left|\frac{3}{7} - \frac{3}{4}\right| = 3\left|\frac{4-7}{28}\right| = \frac{9}{28} \Rightarrow$$ **(D)**]

2. $\frac{dy}{dx} =$ slope fo tangent

$-\frac{1}{t^2} = -\frac{b}{a}$ ∴ $\frac{a}{b} = t^2 > 0$ ⇒ a and b are of same sign.

3. $f'(x) = \sqrt{1-x^4} > 0$ in (–1, 1) ⇒ f is ↑

Now $f(x) + f(-x) = \int_0^x \sqrt{1-t^4}\,dt + \int_0^{-x}\sqrt{1-t^4}\,dt \Rightarrow \int_0^x \sqrt{1-t^4}\,dt + \left(-\int_0^y \sqrt{1-y^4}\,dy\right)$ (t = – y)

= 0 ⇒ f (x) is odd

again $f''(x) = \frac{-4x^3}{2\sqrt{1-x^4}}$ which vanished at x = 0 and changes sign ⇒ (0, 0) is inflection since f is well defined in [–1, 1] ⇒ A, B, C, D]

4. Since intercepts are equal in magnitude but opposite in sign ⇒ $\left.\frac{dy}{dx}\right|_P = 1$

now $\frac{dy}{dx} = x^2 - 5x + 7 = 1$ ⇒ $x^2 - 5x + 6 = 0$ ⇒ x = 2 or 3]

5. $h(x) = \dfrac{\ell n\left(f(x)\,.\,g(x)\right)}{\ell n\, a} = \dfrac{\ell na^{\{a^{|x|}.\text{sgn}\,x\}+[a^{|x|}.\text{sgn}\,x]}}{\ell na}$

$= \left\{a^{|x|}\ \text{sgn}\,x\right\} + \left[a^{|x|}\ \text{sgn}\,x\right] = a^{|x|}\ \ \text{sgn}\ x \qquad (\because\ \{y\} + [y] = y)$

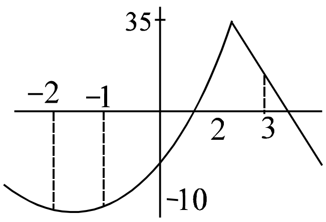
$= \begin{bmatrix} a^{x} & \text{for} & x > 0 \\ 0 & \text{for} & x = 0 \\ -a^{-x} & \text{for} & x < 0 \end{bmatrix} \Rightarrow$ h (x) is an odd function]

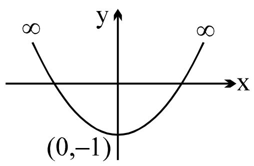
6. $f'(x) = 100\,x^{99} + \cos x$

for $x \in (0,1)$ and $\left(0, \dfrac{\pi}{2}\right)$, cosx and x are both +ve $\Rightarrow \uparrow$

for $x \in \left(\dfrac{\pi}{2}, \pi\right)$, x > 1 hence $100\,x^{99}$ obviously > cosx $\Rightarrow \uparrow$]

7. Note that f (x) is continuous at x= 2 and f is decreasing for (2, 3) and increasing for [–1, 2] .
At x = 2 f has a maxima hence (A) is not correct.]



8. Graph of y = f (x) $\Rightarrow$ (A) and (C) ]

9. If f and g are inverse then (fog)(x) = x
f ' [g(x)] g ' (x) = 1
if f is increasing $\Rightarrow$ f ' > 0 $\Rightarrow$ sign of g ' is also + ve $\Rightarrow$ (A) is correct
If f is decreasing $\Rightarrow$ f ' < 0 $\Rightarrow$ sign of g ' is – ve $\Rightarrow$ (B) is false
since f has an inverse $\Rightarrow$ f is bijective $\Rightarrow$ f is injective $\Rightarrow$ (C) is correct
inverse of a bijective mapping is bijective
$\Rightarrow$ g is also bijective $\Rightarrow$ g is onto $\Rightarrow$ (D) is correct]

10. $f(x) = \ln(1 - \ln x)$
domain (0, e)

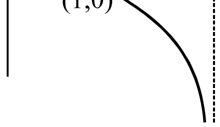
$f'(x) = -\dfrac{1}{(1-\ln x)} \cdot \dfrac{1}{x} < 0 \Rightarrow$ decreasing $\forall$ x in its domain $\Rightarrow$ (A) & (B) are incorrect

$f'(1) = -1 \Rightarrow$ (C) is also incorrect

also $f(1) = 0;\quad \lim\limits_{x\to e^{-1}} f(x) \to -\infty;\ \ \lim\limits_{x\to 0^{+}} f(x) \to \infty$

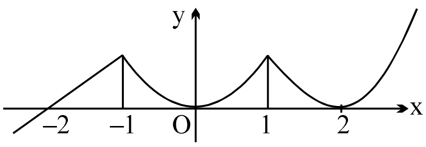
$f''(x) = \dfrac{-\ln x}{x^2(1-\ln x)^2}$

$f''(1) = 0$ which is a point of inflection
graph is as shown



y axis and x = e are two asynmptotes]

11. f is obvious continuous $\forall$ x $\in$ R and not derivable at – 1 and 1
f '(x) changes sign 4 times at – 1, 0, 1, 2
local maxima at 1 and – 1
local minima at x = 0 and 2]



12. Domain is x $\in$ R

Also $f(x) = \left[\cos\left(\tan^{-1}(\sin\theta)\right)\right]^2$ where $\cot\theta = x$

$$= \left[\cos\left(\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)\right]^2 = (\cos\phi)^2 \text{ where } \tan\phi = \frac{1}{\sqrt{1+x^2}}$$

$$= \left(\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right)^2$$

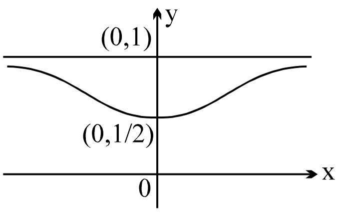
$$g(x) = \frac{1+x^2}{2+x^2} = 1 - \frac{1}{2+x^2}$$

range is $\left[\frac{1}{2}, 1\right)$; $f'(x) = \frac{2x}{(2+x^2)^2}$

hence $f'(0) = 0$

also $\lim_{x\to\infty} f(x) = 1$

hence (B), (C), (D)]



13. Let the tangent line be y = ax + b
The equation for its intersection with the upper parabola is

$x^2 + 1 = ax + b$

$x^2 - ax + (1 - b) = 0$

This has a double root when $a^2 - 4(1 - b) = 0$ or $a^2 + 4b = 4$

For the lower parabola

$ax + b = -x^2$

$x^2 + ax + b = 0$

This has a double root when $a^2 - 4b = 0$

subtract these two equations to get $8b = 4$ or $b = 1/4$

add them to get $2a^2 = 4$ or $a = \pm\sqrt{2}$

The tangent lines are $y = \sqrt{2}x + \frac{1}{2}$ and $y = -\sqrt{2}x + \frac{1}{2}$

14. $f(x) = \int_0^{\pi} \cos t \cos(x - t)dt$ ....(1)

$$= \int_0^{\pi} -\cos t \cdot \cos(x - \pi + t)dt$$

$f(x) = \int_0^{\pi} \cos t \cdot \cos(x + t)dt$ ....(2)

(1) + (2) gives

$$2f(x) = \int_0^{\pi} \cos t\,(2\cos x \cdot \cos t)dt$$

$\therefore \quad f(x) = \cos x \int_0^{\pi} \cos^2 t\,dt = 2\cos x \int_0^{\pi/2} \cos^2 t\,dt$

$f(x) = \frac{\pi \cos x}{2}$ Now verify. Only (A) & (B) are correct.

15. (A) $f(x) = x - \tan^{-1}x$

$f'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \quad \Rightarrow \quad$ f is increasing in (0, 1)

$f(x) > f(0)$ but $f(0) = 0$

$f(x) > 0 \quad \Rightarrow \quad x > \tan^{-1}x$ in (0, 1)

(B) $f(x) = \cos x - 1 + \frac{x^2}{2}$

$f'(x) = -\sin x + x = x - \sin x > 0$ in (0, 1) $\Rightarrow$ **(B) is not correct**

(C) $f(x) = 1 + x\,ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2}$

$$f'(x) = x\left(\frac{1+\frac{1}{2}\cdot\frac{2x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\right) + ln\left(x+\sqrt{1+x^2}\right) - \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{x}{\sqrt{1+x^2}} + ln\left(x+\sqrt{1+x^2}\right) - \frac{x}{\sqrt{1+x^2}} > 0 \quad \forall \ x \in R$$

$\Rightarrow$ (C) is true

(D) $f(x) = x - \frac{x^2}{2} - ln(1+x)$

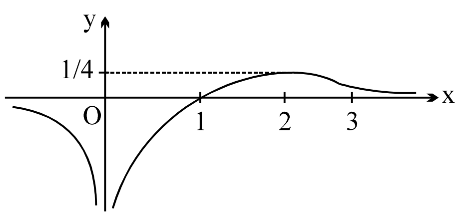
$f'(x) = (1-x) - \frac{1}{1+x} = \frac{(1-x^2)-1}{1+x} = -\frac{x^2}{1+x} < 0 \Rightarrow$ **(D) is correct**

hence f (x) is decreasing in (0, 1)

$\therefore \quad f(x) < f(0)$

$f(x) < 0 \quad \Rightarrow \quad x - \frac{x^2}{2} < ln(1+x)$]

16. $f'(x) = \frac{2-x}{x^3}$ and $f''(x) = \frac{x-3}{x^4}$. Now interpret



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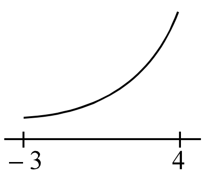


17. (A) f (x) has no relative minimum on (–3, 4)

(B) f (x) is continuous function on [–3, 4]

$\Rightarrow$ f (x) has min. and max. on [–3, 4] by IVT

(C) $f''(x) > 0 \quad \Rightarrow \quad$ f (x) is concave upwards on [–3, 4]

(D) $f(3) = f(4)$

By Rolle's theorem

$\exists\ c \in (3, 4)$, where $f'(c) = 0$

$\Rightarrow\ \exists$ critical point on $[-3, 4]$

18. (A) False, e.g. $f(x) = \sin\sqrt{x}$

(B) True, from IVT

(C) True as $\lim\limits_{x\to\infty} \sin^{-1}\left(1+\frac{1}{x}\right) = \sin^{-1}$(a quantity greater than one) $\Rightarrow$ not defined

(D) True, as the line passes through the centre of the circle.

19.

(A) Let $\ell = \lim\limits_{x\to 0} \dfrac{x\int_0^x e^{t^2}dt}{-(e^x - x - 1)}\left(\frac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{x\int_0^x e^{t^2}dt}{-x^2\left(\frac{e^x - x - 1}{x^2}\right)} = -2\lim\limits_{x\to 0}\dfrac{\int_0^x e^{t^2}dt}{x}\left(\frac{0}{0}\right) = -2\lim\limits_{x\to 0}\dfrac{e^{x^2}}{1} = -2$

(B) $14x^2 - 7xy + y^2 = 2$

$\dfrac{dy}{dx} = \dfrac{28x - 7y}{7x - 2y}$ ....(1)

if $x = 1$ then $14 - 7y + y^2 = 2 \Rightarrow y^2 - 7y + 12 = 0 \Rightarrow y = 3$ or 4

hence L (1, 3) and M (1, 4)

slope of tangent at L = $\dfrac{28-21}{7-6} = 7$; slope of tangent at M = $\dfrac{28-28}{7-8} = 0$

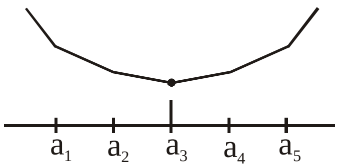
equation of tangent at L and M are

$y - 3 = 7(x - 1) \Rightarrow y = 7x - 4$

and $y - 4 = 0(x - 1) \Rightarrow y = 4$

hence $N = \left(\frac{8}{7}, 4\right) \Rightarrow$ **(C)**

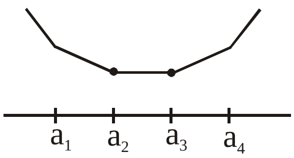
(C) If n is odd then graph of f (x) is



a_3 is the only point where

f (x) has its minimum value

If n is even then graph of f (x) is



From a_2 to a_3 at all values of x, f (x) is minimum.

(D) $2lc + m = (lb^2 + mb)\ \dfrac{-(la^2 + my)}{b - a} = l(b^2 - a^2) + m(b - a) = l(b + a) + m;\ c = \dfrac{a+b}{2}$

20. We have $f'(x) = 5\sin^4 x\cos x - 5\cos^4 x\sin x = 5\sin x\cos x(\sin x - \cos x)(1 + \sin x\cos x)$

$\therefore\ f'(x) = 0$ at $x = \dfrac{\pi}{4}$. Also $f'(0) = f'\left(\dfrac{\pi}{2}\right) = 0$

Hence $\exists$ some $c \in \left(0, \dfrac{\pi}{2}\right)$ for which $f'(c) = 0$ (By Rolle's Theorem) $\Rightarrow$ **(C)** is correct.

Also in $\left(0, \frac{\pi}{4}\right)$ f is decreasing and in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ f is increasing $\Rightarrow$ minimum at $x = \frac{\pi}{4}$

As f (0) = $f\left(\frac{\pi}{2}\right) = 0$ $\Rightarrow$ 2 roots $\Rightarrow$ **(D)** is correct.]

21. $f(x) = \tan^{-1}(x)$ is defined on R and is strictly increasing but do not have its range R]

22. f (0) = 1; f (2) = 2
$f(1^-) = f(1^+) = f(1) = 2$]

23. $f(x) = ln(2 + x) - \frac{2x+2}{x+3}$ is continuous in (–2, ∞)

$$f'(x) = \frac{1}{x+2} - \frac{4}{(x+3)^2} = \frac{(x+3)^2 - 4(x+2)}{(x+2)(x+3)^2}$$

$$= \frac{x^2+2x+1}{(x+2)(x+3)^2} = \frac{(x+1)^2}{(x+2)(x+3)^2} > 0 \qquad (f'(x) = 0 \text{ at } x = -1)$$

$\Rightarrow$ f is increasing in (– 2, ∞)

also $\lim\limits_{x\to-2^+} f(x) \to -\infty$ and $\lim\limits_{x\to\infty} f(x) \to \infty$ $\Rightarrow$ unique root]

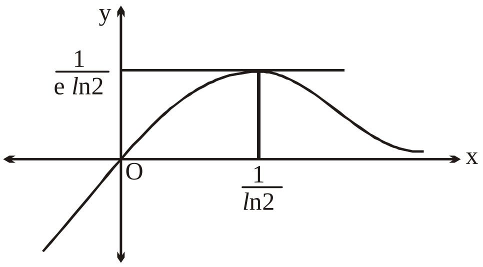
24. Let f (x) = 0 has two roots say $x = r_1$ and $x = r_2$ where $r_1, r_2 \in [a, b]$
$\Rightarrow$ $f(r_1) = f(r_2)$
hence ∃ there must exist some $c \in (r_1, r_2)$ where f ' (c) = 0
but $f'(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$
for $|x| \geq 1$, $f'(x) = (x^6 - x^5) + (x^4 - x^3) + (x^2 - x) + 1 > 0$
for $|x| \leq 1$, $f'(x) = (1 - x) + (x^2 - x^3) + (x^4 - x^5) + x^6 > 0$
hence f ' (x) > 0 for all x
∴ Rolles theorem fails $\Rightarrow$ f (x) = 0 can not have two or more roots.]

25. Consider the example of $f(x) = e^x$ and $f'(x) = e^x$ both increasing]

Paragraph for question nos. 26 to 27

(i) We have $f(x) = x\,2^{-x}$
So, $f'(x) = 2^{-x}(1 - x\,ln2)$
and $f''(x) = 2^{-x}\,ln2\,(x\,ln2 - 2)$

Clearly, f(x) is increasing in $\left(-\infty, \frac{1}{ln2}\right)$ and decreasing in $\left(\frac{1}{ln2}, \infty\right)$.



Graph of $f(x) = x\,2^{-x}$

(ii) . . . . $\in \left(0, \frac{1}{e\,ln 2}\right)$.

(iii) Given $f(x) = x\,2^{-x}$ and $g(x) = \max.\{f(t) : x \le t \le x + 1\}$

As $f(x)$ is increasing in $\left(-\infty, \frac{1}{ln 2}\right)$, hence maximum value of $g(x)$ occurs at $t = x + 1$

$\therefore \quad g(x) = f(x + 1) = (x + 1)\,2^{-(x+1)}$

Let $I = \int\limits_0^{\frac{1}{ln 2}-1} g(x)\,dx = \int \underbrace{(x+1)}_{\substack{I \\ (I.B.P.)}}\underbrace{2^{-(x+1)}}_{II}\,dx$

$$= -\left.\frac{(x+1)\,2^{-(x+1)}}{ln\,2}\right]_0^{\frac{1}{ln\,2}-1} + \frac{1}{ln 2}\int\limits_0^{\frac{1}{ln\,2}-1} 2^{-(x+1)}dx$$

$$= -\left.\frac{(x+1)}{ln 2}\,2^{-(x+1)}\right]_0^{\frac{1}{ln 2}-1} - \left.\frac{1}{ln^2 2}\,2^{-(x+1)}\right]_0^{\frac{1}{ln 2}-1}$$

$$= -\frac{1}{ln 2}\left[\frac{1}{ln 2}\,\frac{1}{e} - \frac{1}{2}\right] - \frac{1}{ln^2 2}\left[\frac{1}{e} - \frac{1}{2}\right] = -\frac{1}{e\,ln^2 2} + \frac{1}{2\,ln 2} - \frac{1}{e\,ln^2 2} + \frac{1}{2\,ln^2 2}$$

$$= \frac{1}{2\,ln^2 2} + \frac{1}{2\,ln 2} - \frac{2}{e\,ln^2 2}$$ **Ans.**]

Paragraph for question nos. 29 to 31

(1) $\lim\limits_{x\to 0^+} x\,ln\left(1+\frac{1}{x}\right) = \lim\limits_{x\to 0^+} \frac{ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty}\right)$

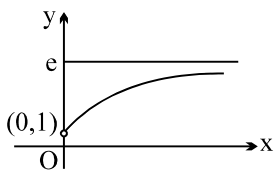
Using L'Hospital's Rule

$l = \lim\limits_{x\to 0} -\left(\frac{1}{x+1} - \frac{1}{x}\right)x^2 = \lim\limits_{x\to 0}\left(\frac{1}{x} - \frac{1}{x+1}\right)\cdot x^2 = \lim\limits_{x\to 0} \frac{1}{x(x+1)}\cdot x^2 = \lim\limits_{x\to 0}\frac{x}{(x+1)} = 0$ Ans.

(2) $\lim\limits_{x\to 0} f(x) = 1$ (can be verified)

$\lim\limits_{x\to\infty} f(x) = e$

Also f is increasing for all $x > 0 \Rightarrow$ (D) (can be verified)



(3) $l = \left(\prod\limits_{k=1}^{n}\left(1+\frac{n}{k}\right)^{k/n}\right)^{1/n}$ {given $f(x) = (1 + 1/x)^x$ and $f(k/n) = \left(1+\frac{n}{k}\right)^{k/n}$ }

taking log,

$ln\,l = \lim\limits_{n\to\infty}\frac{1}{n}\cdot\sum\limits_{k=1}^{n} ln\left(1+\frac{n}{k}\right)^{k/n} = \lim\limits_{n\to\infty}\frac{1}{n}\cdot\sum\limits_{k=1}^{n}\frac{k}{n}\,ln\left(1+\frac{1}{k/n}\right)dx$

$$= \int\limits_0^1 \underbrace{x}_{II}\,\underbrace{ln\left(1+\frac{1}{x}\right)}_{I}dx = \left. ln\left(1+\frac{1}{x}\right)\cdot\frac{x^2}{2}\right]_0^1 + \int\limits_0^1\left(\frac{1}{x} - \frac{1}{x+1}\right)\cdot\frac{x^2}{2}dx$$

$= \left(\frac{1}{2}ln\, 2 - 0\right) + \frac{1}{2}\int_0^1 \frac{x+1-1}{x+1}dx \ = \ \frac{1}{2}ln\, 2 + \frac{1}{2}\left[x - ln(x+1)\right]_0^1$

$= \frac{1}{2}ln\, 2 + \frac{1}{2}\left[(1 - ln\, 2) - 0\right] \ = \ \frac{1}{2}$

$l = \sqrt{e}$

Paragraph for question nos. 32 to 34

$y = \frac{x^2}{x^2-1}$; not defined at $x = \pm 1$

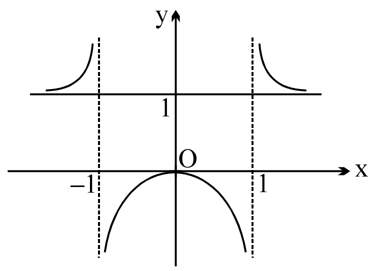
$= 1 + \frac{1}{x^2-1}$; $\qquad y' = -\frac{2x}{(x^2-1)^2}$

$\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad x = 0$ (point of maxima)

as $\quad x \to 1^+, \ y \to \infty \quad ; \quad x \to 1^-, \ y \to -\infty$

|||ly $\quad x \to -1^+, \ y \to -\infty \quad ; \quad x \to -1^-, \ y \to \infty$

The graph of $y = \frac{x^2}{x^2-1}$ is as shown



verify all alternativels from the graph.

Paragraph for question nos. 35 to 37

(i) $a = 1$

$f(x) = 8x^3 + 4x^2 + 2bx + 1$

$f'(x) = 24x^2 + 8x + 2b = 2(12x^2 + 4x + b)$

for increasing function, $\qquad f'(x) \geq 0 \quad \forall \ x \in R$

$\therefore \qquad D \leq 0 \ \Rightarrow \qquad 16 - 48b \leq 0$

$\Rightarrow \qquad b \geq \frac{1}{3} \ \Rightarrow$ **(C)**

(ii) if $b = 1$

$f(x) = 8x^3 + 4ax^2 + 2x + a$

$f'(x) = 24x^2 + 8ax + 2$ $\qquad$ or $\qquad 2(12x^2 + 4ax + 1)$

for non monotonic $f'(x) = 0$ must have distinct roots

hence $D > 0$ i.e. $\quad 16a^2 - 48 > 0 \ \Rightarrow \quad a^2 > 3$;

$\therefore \qquad a > \sqrt{3}$ or $a < -\sqrt{3}$

$\therefore \qquad a \in 2, 3, 4, \ldots\ldots.$

sum = 5050 – 1 = 5049 Ans.

(iii) If x_1, x_2 and x_3 are the roots then $\log_2 x_1 + \log_2 x_2 + \log_2 x_3 = 5$

$\log_2(x_1x_2x_3) = 5$

$x_1x_2x_3 = 32$

$-\frac{a}{8} = 32$

$\Rightarrow \qquad a = -256$ **Ans.**]

38. (A) R; (B) R, S, T ; (C) Q; (D) Q

(A) $I = \int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{(x^2+1)^2-(x^2-1)}{(x^2+1)^2}dx = \int_{\sqrt{2}-1}^{\sqrt{2}+1}\left(1-\frac{(x^2-1)}{(x^2+1)^2}\right)dx = 2 - \underbrace{\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{(x^2-1)}{(x^2+1)^2}dx}_{I_1}$

$I_1 = \int_{1/a}^{a}\frac{(x^2-1)}{(x^2+1)^2}dx$ where $(a = \sqrt{2}+1)$; put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$

$= \int_{a}^{1/a}\frac{\frac{1}{t^2}-1}{\left(\frac{1}{t^2}+1\right)^2}\cdot\left(-\frac{1}{t^2}\right)dt = -\int_{a}^{1/a}\frac{(1-t^2)t^4}{t^4(1+t^2)^2}dt = -\int_{a}^{1/a}\frac{(1-t^2)}{(1+t^2)^2}dt = \int_{a}^{1/a}\frac{t^2-1}{(t^2+1)^2}dt$

$= -\int_{1/a}^{a}\frac{t^2-1}{(t^2+1)^2}dt = -I_1$

$\Rightarrow \quad 2I_1 = 0 \Rightarrow \quad I_1 = 0 \quad \Rightarrow$ 2 is the answer.]

(B) Domain of f (x) is $(0, 1) \cup (1, \infty)$
$ln\, f(x) = 1 \quad \Rightarrow \quad f(x) = e =$ constant
$f'(x) = 0$, for all in $(0, \infty) - \{1\}$

(C) Clearly (1, 0) is the point of intersection of given curves.

Now, $f'(x) = \frac{2^x}{x} + 2^x\ (ln2)\ (lnx)$

$\therefore$ Slope of tangent to the curve f (x) at (1, 0) = $m_1 = 2$

Similarly, $g'(x) = \frac{d}{dx}(e^{2x\, lnx} - 1) = x^{2x}\left(2x \times \frac{1}{x} + 2\, lnx\right)$

$\therefore$ Slope of tangent to the curve g (x) at (1, 0) = $m_2 = 2$
since $m_1 = m_2 = 2$
$\Rightarrow$ Two curves touch each other, so angle between them is 0.
Hence $\cos\theta = \cos 0 = 1$

(D) $3y^2y' - 3y - 3xy' = 0 \quad \Rightarrow \quad y' = \frac{y}{y^2-x}$

$y' = 0 \quad \Rightarrow \quad y = 0$, no real x.
$y' = \infty \quad \Rightarrow \quad y^2 = x \quad \Rightarrow \quad y^3 = 1, y = 1$
The point is (1, 1)

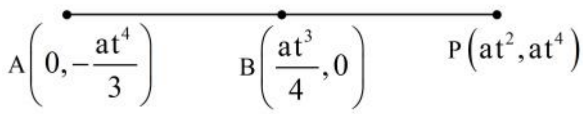
39. $(A)\to R, (B)\to Q, (C)\to P, (D)\to S$

(A) $\frac{dy}{dx} = \frac{4t}{3}$, Tangent is $y - at^4 = \frac{4t}{3}\left(x - at^3\right)$

x-intercept $= \frac{at^3}{4}$

y-intercept $= -\frac{at^4}{3}$

the point of intersection of tangent with the axes are $\left(\frac{at^3}{4}, 0\right)$ and $\left(0, -\frac{at^4}{3}\right)$



P divids AB externaly in 4 : 3

$\therefore \quad \frac{m}{n} = \frac{4}{3} \quad \Rightarrow \quad m = 4 \ \& \ n = 3$

as m & n are coprime to each other

$\therefore \quad m + n = 7$

(B) $\frac{dx}{dy} = e^{\sin y} \cos y :$ slope of normal $= -1$

equation of normal is $x + y = 1$

Area $= \frac{1}{2}$

(C) $y = \frac{1}{x^2} : \frac{dy}{dx} = -\frac{1}{x^3} :$ slope of tangent $= -2$

$y = e^{2-2x} : \frac{dy}{dx} = e^{2-2x}.(-2) :$ slope of tangent $= -2$

$\therefore \tan^{-} = 0$

(D) Length of subtangent $= \left|\frac{y}{y'}\right| = \left|\frac{be^{x/3}}{b\frac{1}{2}e^{x/3}}\right| = 3$

40 (A)$\to$(q),(B)$\to$(r),(C)$\to$(q),(D)$\to$(q)

$y = f(x) + \frac{1}{x}$ so $y > 0$

Now $f(y) f\left(f(y) + \frac{1}{y}\right) = 1$ also $f(x) f(y) = 1$

$\therefore f(x) = f\left(f(y) + \frac{1}{y}\right) = f\left(\frac{1}{f(x)} + \frac{1}{f(x) + 1/x}\right)$

also f(x) is increasing

$\therefore x = \frac{1}{f(x)} + \frac{1}{f(x) + 1/x}$

$\Rightarrow f(x) = \frac{1 \pm \sqrt{5}}{2x}$

now $f(x) = \frac{1+\sqrt{5}}{2x}$ is decreasing so discarding it $f(x) = \frac{1-\sqrt{5}}{2x}$.

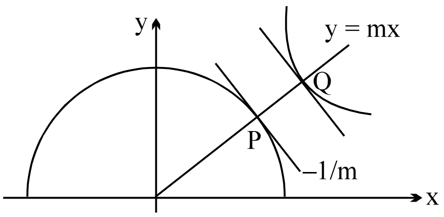
Applications Of Derivatives
Exercise 2(B)

1 Given $S = x^2 + 4xh = 1200$
and $V = x^2h$

$V(x) = \dfrac{x^2(1200 - x^2)}{4x}$; $V(x) = \dfrac{1}{4}(1200x - x^3)$

Put $V'(x) = 0$ gives $x = 20$
If $x = 20, h = 10$
Hence $V_{max.} = x^2h = (400)(10) = 4000$ cubic cm.

2 Note that C_1 is a semicircle and C_2 is a rectangular hyperbola.
PQ will be minimum if the normal at P on the semicircle is also a normal at Q on $xy = 9$
Let the normal at P be $y = mx$ ....(1) $(m > 0)$
solving it with $xy = 9$



$mx^2 = 9 \Rightarrow x = \dfrac{3}{\sqrt{m}}$; $y = \dfrac{9\sqrt{m}}{3}$

$\therefore$ $Q \equiv \left(\dfrac{3}{\sqrt{3}}, 3\sqrt{m}\right)$

differentiating $xy = 9$

$x\dfrac{dy}{dx} + y = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$

$\therefore$ $\left.\dfrac{dy}{dx}\right|_Q = -\dfrac{3\sqrt{m}\cdot\sqrt{m}}{3} = -m$

$\therefore$ tangent at P and Q must be parallel

$\therefore$ $-m = -\dfrac{1}{m} \Rightarrow m^2 = 1 \Rightarrow m = 1$

$\therefore$ normal at P and Q is $y = x$
solving P(1, 1) and Q(3, 3)
$\therefore$ $(PQ)^2 = d^2 = 4 + 4 = 8$ Ans.]

3 The given expression resembles with $(x_1 - x_2)^2 + (y_1 - y_2)^2$, where $y_1 = \dfrac{x_1^2}{20}$ and

$y_2 = \sqrt{(17 - x_2)(x_2 - 13)}$

Thus, we can thing about two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on the curves $x^2 = 20y$ and $(x - 15)^2 + y^2 = 4$ respectively.
Let D be the distance between P_1 and P_2 then the given expression simply represents D^2.
Now, as per the requirements, we have to locate the point on these curves (in the first quadrant) such that the distance between them is minimum.
Since the shortest distance between two curves always occurs along the common normal, it implies that we have to locate a point $P(x_1, y_1)$ on the parabola $x^2 = 20y$ such that normal drawn to parabola at this point passes through (15, 0).

Now, equation of the normal to the parabola at (x_1, y_1) is $\left(y - \dfrac{x_1^2}{20}\right) = \dfrac{-10}{x_1}(x - x_1)$. It should pass through (15, 0).

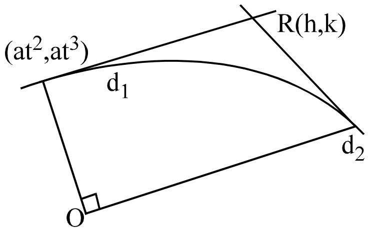
$\Rightarrow$ $x_1^3 + 200x_1 - 3000 = 0$ $\Rightarrow$ $x_1 = 10 \Rightarrow$ $y_1 = 5$

$\Rightarrow \quad D = \sqrt{(10-15)^2+5^2} - 2 = \left(5\sqrt{2}-2\right)$

The minimum value of the given expression is $\left(5\sqrt{2}-2\right)^2 = \left(a\sqrt{2}-b\right)^2$

$\therefore \quad a = 5 \ \& \ b = 2$

4 $x = t^2 \ ; \ y = t^3$



$\frac{dx}{dt} = 2t \ ; \quad \frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{3t}{2}$

$y - t^3 = \frac{3t}{2}(x - t^2) \quad \text{....(1)}$

$2k - 2t^3 = 3th - 3t^3$

$t^3 - 3th + 2k = 0$

$t_1t_2t_3 = -2k \quad (\text{put } t_1t_2 = -1); \qquad \text{hence } t_3 = 2k$

now t_3 must satisfy the equation (1) which gives $4y^2 = 3x - 1$.

Comparing with $ay^2 = bx - 1$, we have a = 4 and b = 3.

5 We have $F(x) = \begin{cases} -2x + \log_{\frac{1}{2}}(k^2-6k+8), & -2 \le x < -1 \\ x^3+3x^2+4x+1, & -1 \le x \le 3 \end{cases}$

Also F(x) is increasing on [–1, 3] because $F'(x) > 0 \ \forall \ x \in [-1, 3]$.

And $F'(x) = -2 \ \forall \ x \in [-2, -1)$, so F(x) is decreasing on [–2,–1).

$\therefore$ If F(x) has smallest value at x = –1, then we must have

$\lim_{h\to 0} F(-1-h) \ge F(-1)$

$\Rightarrow \quad 2 + \log_{1/2}(k^2 - 6k + 8) \ge -1 \quad \Rightarrow \quad \log_{1/2}(k^2 - 6k + 8) \ge -3 \quad \Rightarrow \quad k^2 - 6k + 8 \le 8$

$\Rightarrow \quad k^2 - 6k \le 0 \quad \Rightarrow \quad k \in [0, 6] \qquad \text{....(1)}$

But in order to define $\log_{1/2}(k^2 - 6k + 8)$,

We must have $k^2 - 6k + 8 > 0$

$\Rightarrow \quad (k-2)(k-4) > 0 \quad \Rightarrow \quad k < 2 \text{ or } k > 4 \qquad \text{......(2)}$

$\therefore$ From (1) and (2), we get $k \in [0, 2) \cup (4, 6]$

$\Rightarrow$ Possible integer(s) in the range of k are 0, 1, 5, 6

Hence the sum of all possible positive integer(s) in the range of k = 1 + 5 + 6 = 12 **Ans.**]

6 We have $F(x) = \frac{x^3}{3} + (a-3)x^2 + x - 13$

$\therefore$ For F(x) to have negative point of local minimum, the equation F '(x) = 0 must have two distinct negative roots.

Now, $F'(x) = x^2 + 2(a-3)x + 1$

$\therefore$ Following condition(s) must be satisfied simultaneously.

(i) Discriminant > 0; (ii) Sum of roots < 0 ; (iii) Product of roots > 0

Now, D > 0

$\Rightarrow \quad 4(a-3)^2 > 4 \quad \Rightarrow \quad (a-3)^2 - 1 > 0 \quad \Rightarrow \quad (a-2)(a-4) > 0$

$\therefore \quad a \in (-\infty, 2) \cup (4, \infty) \qquad \text{........... (i)}$

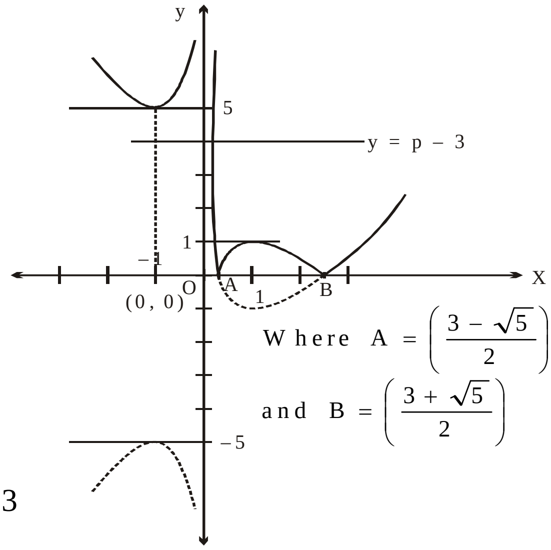
Also $-2(a-3) < 0 \quad \Rightarrow \quad a - 3 > 0 \quad \Rightarrow \quad a > 3 \qquad \text{.......... (ii)}$

And product of root(s) = $1 > 0 \ \forall \ a \in R$

$\therefore \quad (i) \cap (ii) \cap (iii) \quad \Rightarrow \quad a \in (4, \infty) \qquad \text{....(iii)}$

Hence sum of value(s) of a = 5 + 6 + 7 + ... + 100 = 5040 Ans.]

7. Consider $y = x + \frac{1}{x} - 3$



$\Rightarrow \quad \frac{dy}{dx} = 1 - \frac{1}{x^2} = 0$

$\therefore \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 1$ or -1

As $x \to 0^+$, $y \to \infty$ and $x \to 0^-$, $y \to -\infty$

Also roots of $x + \frac{1}{x} - 3 = 0 \Rightarrow x^2 - 3x + 1 = 0$

$$x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

For two distinct solutions either $p - 3 = 0 \Rightarrow p = 3$

or $1 < p - 3 < 5$

$4 < p < 8$

Hence $p \in \{3\} \cup (4, 8)$

$p = \{3, 5, 6, 7\} \quad \Rightarrow \quad$ Sum = 21 **Ans.**]

8 Let $f''(x) = 6a(x-1)$ $(a > 0)$ then $f'(x) = 6a\left(\frac{x^2}{2} - x\right) + b = 3a(x^2 - 2x) + b$.

Now $f'(-1) = 0 \Rightarrow 9a + b = 0 \Rightarrow b = -9a$.

$\therefore \quad f'(x) = 3a(x^2 - 2x - 3) = 0 \Rightarrow x = -1$ and 3.

So $y = f(-1)$ and $y = f(3)$ are two horizontal tangents.

Hence distance between its two horizontal tangents = $|f(3) - f(-1)| = |-22 - 10| = 0032$. **Ans.**]

9 Volume (V) = $\frac{1}{3} A_1 h_1 \quad \Rightarrow h_1 = \frac{3V}{A_1}$

|||ly $h_2 = \frac{3V}{A_2}$, $h_3 = \frac{3V}{A_3}$ and $h_4 = \frac{3V}{A_4}$

So $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4) = (A_1 + A_2 + A_3 + A_4)\left(\frac{3V}{A_1} + \frac{3V}{A_2} + \frac{3V}{A_3} + \frac{3V}{A_4}\right)$

$$= 3V(A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)$$

Now using A.M.-H.M inequality in A_1, A_2, A_3, A_4, we get

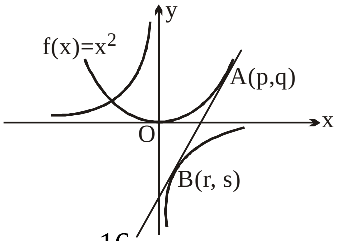
$$\frac{A_1 + A_2 + A_3 + A_4}{4} \geq \frac{4}{\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)}$$

$$\Rightarrow (A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right) \geq 16$$

Hence the minimum value of $(A_1+A_2+A_3+A_4)(h_1+h_2+h_3+h_4) = 3V(16) = 48V = 48 \times 5 = 240$ **Ans.**]

10 $y = x^2$ and $y = -\frac{8}{x}$; $q = p^2$ and $s = -\frac{8}{r}$ ....(1)

Equating $\frac{dy}{dx}$ at A and B, we get



$2p = \frac{8}{r^2}$ ....(1) $\Rightarrow$ $pr^2 = 4$

Now $m_{AB} = \frac{q-s}{p-r}$ $\Rightarrow$ $2p = \frac{p^2 + \frac{8}{r}}{p-r}$ $\Rightarrow$ $p^2 = 2pr + \frac{8}{r}$ $\Rightarrow$ $p^2 = \frac{16}{r}$

$\Rightarrow$ $\frac{16}{r^4} = \frac{16}{r}$ $\Rightarrow$ $r = 1$ $(r \neq 0)$ $\Rightarrow$ $p = 4$

$\therefore$ $r = 1$, $p = 1$

Hence $p + r = 5$

11 $x = 0$ and $x = 1$]

12 $y = x^2$

$\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = 10$m/sec.

$\tan\theta = \frac{x^2}{x} = x$

$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$

$\frac{d\theta}{dt} = 10 \times \cos^2\theta = 10 \times \frac{1}{10} = 1$ { at x = 3m }

13 $3x^2 - 2y\frac{dy}{dx} = 0$ $\Rightarrow$ $\frac{dy}{dx} = \frac{3x^2}{2y}$

slope of tangent at $(4a^2, 8a^3) = \frac{3(16a^4)}{2(8a^3)} = 3a$

let this tangent at this point also cuts the curve at $(4b^2, 8b^3)$ and normal at this point slope of

normal at $(4b^2, 8b^3) = -\frac{1}{3b}$.

$\Rightarrow$ $3a = -\frac{1}{3b}$ $\Rightarrow$ $ab = -\frac{1}{9}$ ......(i)

slope of line = $\frac{8a^3 - 8b^3}{4a^2 - 4b^2} = \frac{2(a^3 - b^3)}{(a^2 - b^2)} = \frac{2(a^2 + b^2 + ab)}{a + b}$

= 3a [it is equal to slope of target]

$\Rightarrow$ $2a^2 + 2b^2 + 2ab = 3a^2 + 3ab$

$\Rightarrow\ 2b^2 = a^2 + ab \quad \Rightarrow\ \frac{2}{81a^2} = \frac{a^2-1}{9}$

$2 = 81a^4 - 9a^2$

$\Rightarrow\ 81a^4 - 9a^2 - 2 = 0$

$81a^4 - 18a^2 + 9a^2 - 2 = 0$

$9a^2(9a^2 - 2) + (9a^2 - 2) = 0$

$\Rightarrow\ (9a^2 - 2)(9a^2 + 1) = 0$

$9a^2 = 2$

14. Let $x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow\quad r^2 = x^2 + y^2;\quad \tan\theta = \frac{y}{x} \qquad \theta \in (0, \pi/2)$

$$N = \frac{r^2}{r^2[\cos^2\theta + \sin\theta\cos\theta + 4\sin^2\theta]} = \frac{r^2}{(1+\cos 2\theta) + \sin 2\theta + 4(1-\cos 2\theta)} = \frac{2}{5+\sin 2\theta + 3\cos 2\theta}$$

$$N_{max} = \frac{2}{5-\sqrt{10}} = \frac{2}{15}\left(5+\sqrt{10}\right) = M$$

$$N_{max} = \frac{2}{5+\sqrt{10}} = \frac{2}{15}\left(5-\sqrt{10}\right) = m$$

$$A = \frac{M+m}{2} = \frac{2\cdot 10}{15\cdot 2} = \frac{2}{3} \qquad \Rightarrow \qquad 2007 \times \frac{2}{3} = 1338 \text{ Ans.]}$$

15. $\frac{f(3)}{f(6)} = \frac{2^{3k}+9}{2^{6k}+9} = \frac{1}{3};\qquad f(9) - f(3) = (2^{9k} + 9) - (2^{3k} + 9) = 2^{9k} - 2^{3k} \qquad ....(1)$

$3(2^{3k} + 9) = 2^{6k} + 9$

$\Rightarrow\quad 2^{6k} - 3(2^{3k}) - 18 = 0$

$2^{3k} = y$

$y^2 - 3y - 18 = 0$

$(y - 6)(y + 3) = 0$

$y = 6;\quad y = -3$ (rejected)

$2^{3k} = 6$

now $\quad f(9) - f(3) = 2^{9k} - 2^{3k} \quad$ { from (1) }

$= (2^{3k})^3 - 2^{3k}$

$= 6^3 - 6 = 210$

hence $\quad N = 210 = 2\cdot 3\cdot 5\cdot 7$

Total number of divisor $= 2\cdot 2\cdot 2\cdot 2 = 16$

number of divisors which are composite = 16 – (1, 2, 3, 5, 7) = 11 Ans.]

16. $f(-3) = f(3) = 2 \qquad$ [f(x) is an even function, $\quad\therefore\quad f(-x) = f(x)$]

again $\quad f(-1) = f(1) = -3$

$\therefore\quad 2\,|f(-1)| = 2\,|f(1)| = 2\,|-3| = 6$

from the graph, $\qquad -3 < f\left(\frac{7}{8}\right) < -2$

$\therefore\quad \left[f\left(\frac{7}{8}\right)\right] = -3$

$f(0) = 0 \qquad$ (obviously from the graph)

$\cos^{-1}(f(-2)) = \cos^{-1}(f(2)) = \cos^{-1}(1) = 0$

$f(-7) = f(-7+8) = f(1) = -3$ [f (x) has period 8]

$f(20) = f(4+16) = f(4) = 3$ [$f(nT + x) = f(x)$]

sum = 2 + 6 – 3 + 0 + 0 – 3 + 3

$\therefore$ sum = 5

17 We have $f(x) = (b^2 - 3b + 2)(\cos^2x - \sin^2x) + (b-1)x + \sin 2$

$\therefore$ $f'(x) = (b-1)(b-2)(-2\sin 2x) + (b-1)$

Now, $f'(x) \neq 0$ for every $x \in R$,

so $(b-1)\left(1 - 2(b-2)\sin 2x\right) \neq 0 \ \forall \ x \in R$

$\therefore$ $b \neq 1$

Also, $\left|\dfrac{1}{2(b-2)}\right| > 1 \Rightarrow b \in \left(\dfrac{3}{2}, 2\right) \cup \left(2, \dfrac{5}{2}\right)$

Now, when b = 2, $f(x) = x + \sin 2 \Rightarrow f'(x) = 1\ (\neq 0)$.

Hence, $b \in \left(\dfrac{3}{2}, \dfrac{5}{2}\right) \Rightarrow b_1 = \dfrac{3}{2}$ and $b_2 = \dfrac{5}{2}$

$\Rightarrow (b_1 + b_2) = \dfrac{3}{2} + \dfrac{5}{2} = \dfrac{8}{2} = 4$

18. Let $x = r\cos\theta$ and $y = r\sin\theta$

$E = (x+5)(y+5) = (r\cos\theta + 5)(r\sin\theta + 5) = r^2\sin\theta\cos\theta + 5r(\cos\theta + \sin\theta) + 25$

Now put $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + xy + y^2 = 3$

$\Rightarrow r^2\cos^2\theta + r^2\sin\theta\cos\theta + r^2\sin^2\theta = 3$

$\Rightarrow r^2(1 + \sin\theta\cos\theta) = 3 \Rightarrow r^2 = \dfrac{3}{1+\sin\theta\cos\theta} = \dfrac{6}{2+\sin 2\theta}$

hence $r^2]_{min.} = 2 + \sin 2\theta]_{max.} = 3$ occurs at $\sin 2\theta = 1 \Rightarrow 2\theta = \dfrac{\pi}{2}$ or $\dfrac{5\pi}{2}$ i.e. $\dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$

Hence $E = \dfrac{r^2}{2}(\sin 2\theta) + 5r(\cos\theta + \sin\theta) + 25$

put $r^2 = 2$ and $\theta = \dfrac{\pi}{4} \Rightarrow E = 1 + 5\sqrt{2}\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\right) + 25 = 36$

put $r^2 = 2$ and $\theta = \dfrac{5\pi}{4} \Rightarrow E = 1 + 5\sqrt{2}\left(-\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\right) + 25 = 16$

hence minimum value of E is 16

19. Using LMVT for f in [1, 2]

$\forall\ c \in (1, 2)\quad \dfrac{f(2) - f(1)}{2-1} = f'(c) \leq 2$

$f(2) - f(1) \leq 2 \quad \Rightarrow \quad f(2) \leq 4$ ....(1)

again using LMVT in [2, 4]

$\forall\ d \in (2, 4)\quad \dfrac{f(4) - f(2)}{4-2} = f'(d) \leq 2$

$\therefore\quad f(4) - f(2) \leq 4$

$8 - f(2) \le 4$

$4 \le f(2) \quad \Rightarrow \quad f(2) \ge 4 \quad ....(2)$

from (1) and (2) $\quad f(2) = 4$

20. Let x tree be added then

$P(x) = (x + 50)(800 - 10x)$

now $\quad P'(x) = 0 \quad \Rightarrow \quad x = 15$

APPLICATIONS OF DERIVATIVES
EXERCISE 2(C)

1. $y = ax^3 + bx^2 + cx + 5$

It has repeated root (-2)

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

$y(-2) = 0$

$\therefore \quad -8a + 4b - 2c + 5 = 0$

$y'(-2) = 0$

$\therefore \quad 12a - 4b + c = 0$

$y'(0) = 3$

$\therefore \quad c = 3$

$\therefore \quad 12a - 4b + 3 = 0$ and $-8a + 4b - 1 = 0$

$\therefore \quad a = -\frac{1}{2}$ and $b = -\frac{3}{4}, c = 3$

2. $xy = 4, x^2 + y^2 = 8$

$m_1 = -\frac{y}{x}, m_2 = -\frac{x}{y}$

Clearly curves intersect at $\left(\pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$

Here , $m_1 = m_2$

$\therefore$ Curves touch each other.

3. $y = \cos(x + y)$

$\therefore \quad \frac{dy}{dx} = -\sin(x + y)\left[1 + \frac{dy}{dx}\right]$

$\therefore \quad \frac{dy}{dx} = \frac{-\sin(x + y)}{1 + \sin(x + y)} = -\frac{1}{2}$ .....[Given]

$\therefore \quad 2\sin(x + y) = 1 + 1\sin(x + y)$

$\therefore \quad \sin(x + y) = 1$

$\therefore \quad x + y = (4n + 1)\frac{\pi}{2}$ and $y = \cos\left(\frac{(4n + 1)\pi}{2}\right) = 0$

$\therefore \quad x = \frac{(4n + 1)\pi}{2}$

But $-2\pi \le x \le 2\pi$

$\therefore \quad x = \frac{-3\pi}{2} \quad$ or $\quad \frac{\pi}{2}$

$\therefore$ Equation of tangents are $x + 2y = \frac{\pi}{2}$ and $x + 2y = \frac{-3\pi}{2}$

4. $x^2 = 4y$

$\frac{dy}{dx} = \frac{x}{2}$

$\therefore \quad -\frac{dx}{dy} = -\frac{2}{x} = m_N$

$\therefore \quad m_N = -2$

$y - 2 = -2(x - 1)$

$\therefore \quad 2x + y = 4$

5. Given curve is $y = \sin x$ ...(1)

$\therefore \quad \frac{dy}{dx} = \cos x$

Equation ot tangent at (x, y) is $Y - y = \frac{dy}{dx}(X - x)$

or $Y - y = \cos x (X - x)$ ...(2)

Since tangent are drawn fro origin then origin (0,0) lies on (2)

$\therefore \quad 0 - y = \cos(0 - x) \quad$ or $\quad \cos x = \frac{y}{x}$ ...(2)

From (1) and (2),

$\sin^2 x + \cos^2 x = y^2 + \frac{y^2}{x^2}$

$\Rightarrow \quad 1 = \frac{x^2 y^2 + y^2}{x^2} \quad \Rightarrow \quad x^2 y^2 = x^2 - y^2.$

6. $\tan\theta = n a^{1-n} x^{n-1}$

$\therefore \quad SN = y \tan\theta$

$= na^{2-2n} x^{2n-1}$

$=$ Constant

$\therefore \quad n = \frac{1}{2}$

7. Given $x^{m+n} = a^{m-n} . y^{2n}$ ...(1)

Taking logarithm of both sides, we get $(m+n)\ln x = (m-n)\ln a + 2n \ln y$

Differentiating of both sides w.r.t. x, we get $\frac{(m+n)}{x} = 0 + \frac{2n}{y}\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{(m+n)}{2n}\cdot\frac{y}{x}$

Now $\frac{(\text{Sub-tangent})^m}{(\text{Sub-normal})^n} = \frac{\left(y\frac{dx}{dy}\right)^m}{\left(y\frac{dy}{dx}\right)^n} = \frac{y^{m-n}}{\left(\frac{dy}{dx}\right)^{m+n}}$

$$= \frac{y^{m-n}}{\left\{\frac{(m+n)}{2n}\cdot\frac{y}{x}\right\}^{m+n}} = \frac{x^{m+n}}{\left(\frac{m+n}{2n}\right)^{m+n}.y^{2n}} = \frac{a^{m-n}}{\left(\frac{m+n}{2n}\right)^{m+n}} \qquad \{\text{from (1)}\}$$

$(\text{Sub-tangent})^m \propto (\text{Sub-normal})^n$

8. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$

$f'(x) = 4x^3 - 24x^2 + 44x - 24 > 0$

$\therefore \quad x^3 - 6x^2 + 11x - 6 > 0$

$\therefore \quad (x-1)(x-2)(x-3) > 0 \qquad \therefore \quad x \in (1,2)\cup(3,\infty)$

9. $f(x) = x - 2\sin x$

$f'(x) = 1 - 2\cos x > 0$

$\therefore \quad \cos x < \frac{1}{2}$ in $0 \le x \le 2\pi$

$\therefore \quad x \in \left(\frac{\pi}{3},\frac{5\pi}{3}\right)$.

10. $f(x) = 2x^2 - \ell n|x|$

$\therefore \quad f'(x) = 4x - \frac{1}{x}$

$= \frac{(2x+1)(2x-1)}{x}$

M.I. in $\left(-\frac{1}{2},0\right)\cup\left(\frac{1}{2},\infty\right)$ & M.D. in $\left(-\infty,-\frac{1}{2}\right)\cup\left(0,\frac{1}{2}\right)$

11. $f(x) = \sin x + \cos x \qquad x \in [0,2\pi]$

$\therefore \quad f'(x) = \cos x - \sin x$

$\therefore \quad f(x)$ is M I in $\left[\left(0,\frac{\pi}{4}\right)\cup\left(\frac{5\pi}{4},2\pi\right)\right]$ & MD in $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$.

12. $g(x) = f(x) + f(1-x) \qquad x \in [0,1]$

$g'(x) = f'(x) - f'(1-x)$

$f''(x) < 0$

$\therefore \quad f'(x) > f'(1-x)$

$\Rightarrow \quad x < 1-x$

$\Rightarrow \quad$ in $x \in \left[0,\frac{1}{2}\right) \quad g'(x) > 0 \quad \& \quad g'(x) < 0$ in $x \in \left(\frac{1}{2},1\right]$

13. TPT $\ell n(1+x) < 0 \;\; ; x > 0$

i.e. $\quad x > \ell n(1+x)$

Let $\quad f(x) = x - \ell n(1+x)$

$\therefore \quad f'(x) = 1 - \frac{1}{1+x}$

$$= \frac{x}{1+x} > 0 \qquad \forall \; x > 0.$$

Now, $\quad f(0) = 0$

$\therefore \quad f(x) > 0 \qquad \forall \, x > 0$

$\therefore \quad x > \ell n(1+x)$

14. $ax^2 + \frac{b}{x} \geq c$

Let $\quad f(x) = ax^2 + \frac{b}{x}$

$\therefore \quad f'(x) = 2ax - \frac{b}{x^2}$

$$= \frac{2ax^3 - b}{x^2} = 0 \text{ at point of minima.}$$

as $2ax^3 - b$ is an increasing function $\forall \; a > 0$.

$\therefore \quad x = \left(\frac{b}{2a}\right)^{\frac{1}{3}}$

$\therefore \quad ax^2 + \frac{b}{x} = a\left(\frac{b^2}{4a^2}\right)^{\frac{1}{3}} + b \times \left(\frac{2a}{b}\right)^{\frac{1}{3}}$

$$= \frac{a^{\frac{1}{3}} b^{\frac{2}{3}}}{4^{\frac{1}{3}}} + 2^{\frac{1}{3}} b^{\frac{2}{3}} a^{\frac{1}{3}}$$

$$= \frac{3 a^{\frac{1}{3}} b^{\frac{2}{3}}}{4^{\frac{1}{3}}} \geq c$$

$\therefore$ $27ab^2 \geq 4c^3$ [Cubing both sides]
as a,b,c are positive.

15. $0 < x_1 < x_2 < \frac{®}{2}$

T.P. $\frac{\tan x_2}{\tan x_1} > \frac{x_1}{x_2}$

i.e. T.P. $x_2 \tan x_2 > x_1 \tan x_1$

Let $f(x) = x \tan x$

$\therefore$ $f'(x) = \tan x + x \sec^2 x > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$

as $\tan x$ & $x \sec^2 x$ are positive

16. P.T. $x > \sin x > x - \frac{x^3}{6}$ for $0 < x \leq \frac{\pi}{2}$

Let $f(x) = x - \sin x$

$\therefore$ $f'(x) = 1 - \cos x > 0$ for $0 < x \leq \frac{\pi}{2}$

$\therefore$ $x > \sin x$

Let $g(x) = \sin x - x + \frac{x^3}{6}$

$\therefore$ $g'(x) = \cos x - 1 + \frac{x^2}{2}$

$\therefore$ $g''(x) = x - \sin x > 0$ for $0 < x \leq \frac{\pi}{2}$ as shown

$\therefore$ $g'(x) = 0$ for $x = 0$ & increases as x increases

$\therefore$ $g'(x) > 0$ for $0 < x \leq \frac{\pi}{2}$

$g(x) = 0$ for $x = 0$

$\therefore$ $g(x) > 0$ for $0 < x \leq \frac{\pi}{2}$

$\therefore \quad \sin x > x - \frac{x^3}{6}$

17. $f(x) = \frac{1}{8}\ell n\, x - bx + x^2$

$\therefore \quad f'(x) = 2x - b + \frac{1}{8x} = 0$

$\therefore \quad 2x^2 - bx + 1 = 0$

$\therefore \quad x = \frac{b \pm \sqrt{b^2 - 4}}{4} \quad$ If $b > 2$

$f''(x) = 2 - \frac{1}{8x} \qquad > 0$ if $x = \frac{b + \sqrt{b^2 - 4}}{4}$ and

< 0 if $x = \frac{b - \sqrt{b^2 - 4}}{4}$

$\therefore$ It has maxima at $x = \frac{b - \sqrt{b^2 - 4}}{4}$ and

minima at $x = \frac{b + \sqrt{b^2 - 4}}{4}$.

18. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$x = 0 \quad \Rightarrow \quad y = b\,\mathrm{cosec}\,\theta$ and

$y = 0 \quad \Rightarrow \quad x = a\sec\theta$

Now, length of intercept $= \sqrt{(a\sec\theta - 0)^2 + (b\,\mathrm{cosec}\,\theta - 0)^2} = \sqrt{a^2\sec^2\theta + b^2\,\mathrm{cosec}^2\theta}$

Which has minimum value $a + b$ is

$a^2\sec^2 x + b^2\,\mathrm{cosec}^2 x = f(x)$

$f'(x) = 2a^2\sec^2 x\tan x - 2b^2\,\mathrm{cosec}^2 x\cot x = 0$

$f(x)$ can have minima only as maxima is ∞.

$\therefore \quad a^2\sec^2 x\tan x = b^2\,\mathrm{cosec}^2 x\cot x$

$\therefore \quad a^2\frac{\sin x}{\cos^3 x} = b^2\frac{\cos x}{\sin^3 x}$

$\therefore \quad \tan^4 x = \frac{b^2}{a^2}$

$\therefore \quad \tan^2 x = \left|\frac{b}{a}\right|, \ \cot^2 x = \left|\frac{a}{b}\right|$

For $a, b > 0$

$$\sec^2 x = \frac{a+b}{a}, \operatorname{cosec}^2 x = \frac{a+b}{b}$$

$\therefore$ $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$

$$= a^2 + ab + ab + b^2 = (a+b)^2$$

19. $S = \pi r\sqrt{r^2 + b^2} + \pi r^2$

$$= \pi r^2 \operatorname{cosec}\alpha + \pi r^2 = \text{constant}$$

$V = \frac{1}{3}\pi r^3 \cot\alpha$ $\therefore$ $r = \sqrt{\frac{S}{\pi(\operatorname{cosec}\alpha + 1)}}$

$$= \frac{1}{3}\frac{\pi S^{\frac{3}{2}} \cot\alpha}{\pi^{\frac{3}{2}}(\operatorname{cosec}\alpha + 1)^{\frac{3}{2}}}$$

$$V = \sqrt{\frac{S^3}{9\pi}} \times \frac{(\cos\alpha)\sqrt{\sin\alpha}}{(1+\sin\alpha)^{\frac{3}{2}}}$$

$$= \frac{(\cos\alpha)\sqrt{\sin\alpha}(1-\sin\alpha)^{\frac{3}{2}}}{\cos^3\alpha} \times \sqrt{\frac{S^3}{9\pi}}$$

$$= \sqrt{\sin\alpha}(1-\sin\alpha)^{\frac{3}{2}} \sec^2\alpha \times \sqrt{\frac{S^3}{9\pi}}$$

$$\therefore \quad \frac{dV}{d\alpha} = \sqrt{\frac{S^3}{9\pi}}\left[\frac{\cos\alpha(1-\sin\alpha)^{\frac{3}{2}}\sec^2\alpha + 2\sqrt{\sin\alpha}(1-\sin\alpha)^{\frac{3}{2}}\sec^2\alpha\tan\alpha}{2\sqrt{\sin\alpha}} - \frac{3\sqrt{\sin\alpha}\sec^2\alpha(1-\sin\alpha)^{\frac{1}{2}} \times \cos\alpha}{2}\right] = 0$$

$\therefore$ $1 - \sin\alpha = 0$

$\Rightarrow$ $\sin\alpha = 1$ rejected as $\alpha \in \left(0, \frac{\pi}{2}\right)$

OR

$$\therefore \quad \frac{\cos\alpha(1-\sin\alpha)\sec^2\alpha + 2\sqrt{\sin\alpha}(1-\sin\alpha)\sec^2\alpha\tan\alpha}{2\sqrt{\sin\alpha}}$$

$$= 3\frac{\sqrt{\sin\alpha}\sec^2\alpha\cos\alpha}{2}$$

$$\therefore \quad \frac{\cos\alpha(1-\sin\alpha) + 2\sin\alpha(1-\sin\alpha)\tan\alpha}{2} = \frac{3\sin\alpha\cos\alpha}{2}$$

$$\therefore \quad \cos\alpha + 4\sin\alpha\tan\alpha - 4\sin^2\alpha\tan\alpha = 4\sin\alpha\cos\alpha$$

$\therefore \quad \cos^2\alpha + 4\sin^2\alpha - 4\sin^3\alpha = 4\sin\alpha\left(1-\sin^2\alpha\right)$

$\therefore \quad 3 + 3\sin^2\alpha - 4\sin^3\alpha = 4\sin\alpha - 4\sin^3\alpha$

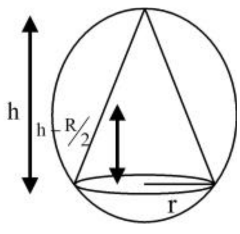
$\therefore \quad 3\sin^2\alpha - 4\sin\alpha + 3 = 0$

$\Rightarrow \quad \sin\alpha = \frac{1}{3}$ hence proved.

20. Refer (Q.21) Assifnment

Ans. $\frac{1}{3}$m

21.



$r^2 + (h-R)^2 = R^2$

$\therefore \quad r^2 = 2Rh - h^2$

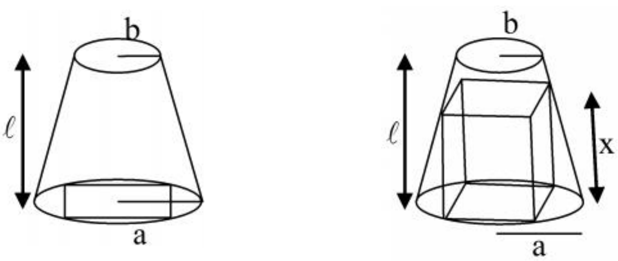
$\therefore \quad V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(2Rh^2 - h^3\right)$

$\therefore \quad \frac{dV}{dh} = \frac{1}{3}\left(4Rh - 3h^3\right) = 0$

$\therefore \quad h = 0$ Rejected or $h = \frac{4R}{3}$

22.



Where h is the height of cut off conical portion

$$\frac{b}{h} = \frac{a}{\ell + h} = \frac{s}{\sqrt{2}\left(h + \ell - x\right)}$$

$$\therefore \quad s = \frac{\sqrt{2}\, b\left(h + \ell - x\right)}{h}$$

$$V = \frac{2b^2\left(h + \ell - x\right)^2}{h^2} \times x$$

$$= \frac{2b^2}{h^2}\left(h^2x + \ell^2x + x^3 + 2h\ell x - 2hx^2 - 2\ell x^2\right)$$

$\therefore \quad \frac{dV}{dx} = \frac{2b^2}{h^2}\left(h^2 + \ell^2 + 2h\ell + 3x^2 - 4hx - 4\ell x\right) = 0$

$\therefore \quad 3x^2 - 4(h-\ell)x + (h+\ell)^2 = 0$

$\therefore \quad x = h + \ell$ rejected or

$x = \frac{(h+\ell)}{3}$

Now, $b\ell + bh = ah$

$\therefore \quad h = \frac{b\ell}{a-b}$

$\therefore \quad x = \frac{1}{3}\left(\frac{b\ell}{a-b} + \ell\right) = \frac{a\ell}{3(a-b)}$

23. Profit = SP − CP

$$= \frac{n}{2}(100-n) - \frac{n^2}{4} - 35n - 25$$

$P = -\frac{3n^2}{4} + 15n - 25$

$\therefore \quad \frac{dP}{dn} = -\frac{3n}{2} + 15 = 0$

$\therefore \quad n = 10$

24. $R = (500 - x)(300 + x)$ $\qquad$ x = Surcharge

$\therefore \quad \frac{dR}{dx} = -300 - x + 500 - x = 0$

$\therefore \quad x = 100$

25. $f(x) = |2x-1| - 2|x-1| - 3$

$\therefore \quad f(x) = 1 - 2x + 2x - 2 - 3 = -4, \quad x \le \frac{1}{2}$

$= 2x - 1 + 2x - 2 - 3 = 4x - 6, \quad \frac{1}{2} < x \le 1$

$= 2x - 1 - 2x + 2 - 3 = -2, \quad x > 1$

$\therefore$ Range of $f(x)$ is $[-4, -2]$

26. $x + y = 20$ $\qquad$ $y = 20 - x$

$x^3y^2 = \max = P$

$\therefore \quad P = x^3(20-x)^2$

$= x^5 - 40x^4 + 400x^3$

$\therefore \quad \frac{dP}{dx} = 5x^4 - 160x^3 + 1200x^2 = 0$

$\therefore \quad x = 0 \quad \Rightarrow$ Rejected or

$5x^2 - 160x + 1200 = 0$

$\Rightarrow \quad x^2 - 32x + 240 = 0$

$\therefore \quad x = \frac{32 \pm \sqrt{64}}{2} = 20 \text{ or } 12$

$x = 20$ rejected

$\therefore \quad x = 12 \ \& \ y = 8$

27. $f(a)g(b) - f(b)g(a) = (b-a)\left[f(a)g'(c) - g(a)f'(c)\right]$

29. $f(x) = x(x+3)e^{-x/2} \qquad [-3, 0]$

$f'(x) = (2x+3)e^{-x/2} - \frac{1}{2}x(x+3)e^{-x/2} = 0$

at $x = c$

$\therefore \quad 2(2x+3) = x^2 + 3x$

$\therefore \quad x^2 - x - 6 = 0$

$\therefore \quad x = -2$ i.e. as $c < 0$

i.e. $\quad c = -2$

30. $f(x) = x(x-1)(x-2)$

$\therefore \quad f(0) = 0, f\left(\frac{1}{2}\right) = \frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2} = \frac{3}{8}$

$\therefore \quad f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$

$$f'(c) = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}$$

$\therefore \quad \frac{3}{4} = 3c^2 - 6c + 2$

$\therefore \quad 12c^2 - 24c + 5 = 0$

$\therefore \quad c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{24 \pm \sqrt{336}}{24} = \frac{6 - \sqrt{21}}{6} \quad$ as $c < \frac{1}{2}$

APPLICATIONS OF DERIVATIVES
EXERCISE 3

1. Since the curve $y = ax^3 + bx^2 + cx + 5$ touches x-axis at P(-2,0) then x-axis is the tangent at (-2,0). The curve meets y-axis in (0,5). We have $\frac{dy}{dx} = 3ax^2 + 2bx + c$

$\therefore \quad \left.\frac{dy}{dx}\right|_{(0,5)} = 0 + 0 + c = 3$ (given)

$\therefore \quad c = 3$ ...(1)

and $\left.\frac{dy}{dx}\right|_{(-2,0)} = 0 \quad \Rightarrow \quad 12a - 4b + c = 0$

$\Rightarrow \quad 12a - 4b + 3 = 0$ {From (1)} ...(2)

and $(-2,0)$ lies on the curve then

$0 = -8a + 4b - 2c + 5 \quad \Rightarrow \quad 0 = -8a + 4b - 1$ $(\because c = 3)$

$\Rightarrow \quad 8a - 4b + 1 = 0$ ...(3)

From (2) and (3) we get $a = -\frac{1}{2}, b = -\frac{3}{4}$ and $c = 3$.`

2. $x^3 - 3xy^2 + 2 = 0$

$\therefore \quad 3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$\therefore \quad x^2 - y^2 = 2xy\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = m_1$

$3x^2y - y^3 + 2 = 0$

$\therefore \quad 6xy + 3x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx}\left(x^2 - y^2\right) = -2xy$

$\therefore \quad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = m_2$

$m_1 m_2 = -1$

3. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$\therefore \quad \frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}$

At (a, b)

$\frac{dy}{dx} = -\frac{b}{a}$

$\therefore \quad b(x-a) + a(y-b) = 0$

$\therefore \quad bx + ay = 2ab$

$\therefore \quad \frac{x}{a} + \frac{y}{a} = 2.$

4. $ay^2 = (x+b)^3$

$2ay\frac{dy}{dx} = 3(x+b)^2$

$\frac{dy}{dx} = \frac{3(x+b)^2}{2ay}$

$ST = \frac{2ay^2}{3(x+b)^2}$, $SN = \frac{3(x+b)^2}{2a}$

$\frac{(ST)^2}{SN} = \frac{4a^2y^4}{3(x+b)^4} \times \frac{2a}{3(x+b)^2}$

$= \frac{8a(ay^2)^2}{27(ay^2)^2} = \frac{8a}{27}.$

5. $y = (1+x)^y + \sin^{-1}(\sin^2 x)$

if $\quad x = 0 \;;\; y = 1$

now $\quad y = e^{y\, ln(1+x)} + \sin^{-1}(\sin^2 x)$

differentiating $\quad \frac{dy}{dx} = (1+x)^y\left[\frac{y}{1+x} + \ln(1+x)\frac{dy}{dx}\right] + \frac{\sin 2x}{\sqrt{1-\sin^4 x}}$

put $x = 0$ and $y = 1$

$\frac{dy}{dx} = 1$

slope of normal = – 1

$y - 1 = -1(x - 0)$

$x + y - 1 = 0$

6. $2y = -5x^5 + 10x^3 - x - 6$

$\therefore \quad \frac{dy}{dx} = -\frac{25}{2}x^4 + 15x^2 - \frac{1}{2}$

At $x = 0, m_N = -\frac{dx}{dy} = 2$

$\therefore \quad y + 3 = 2x$

$\therefore \quad y = 2x - 3$

$\therefore \quad 4x - 6 = -5x^5 + 10x^3 - x - 6$

$\therefore \quad 5x^5 - 10x^3 + 5x = 0$

$\therefore \quad x\left(x^4 - 2x + 1\right) = 0$

$\therefore \quad x\left(x+1\right)^2\left(x-1\right)^2 = 0$

$\therefore \quad x + 1 = 0 \quad$ or $\quad x - 1 = 0$

$\therefore \quad x = \pm 1$

$x = 1 \quad \Rightarrow \quad y = -1$

$x = -1 \quad \Rightarrow \quad y = -5$

$m_{x=1} = 2 = m_{x=-1}$

$\therefore$ Equation of tangents are

$y + 5 = 2\left(x+1\right)$ and $y + 1 = 2\left(x-1\right)$

i.e. $\quad y = 2x - 3 \quad$ i.e. the normal becomes the tangent.

7. $x^3 + y^3 = a^3$

$\therefore \quad 3x^2 + 3y^2\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{x^2}{y^2}$

$\therefore \quad \frac{y - y_1}{x - x_1} = -\frac{x_1^2}{y_1^2}$

$\therefore \quad y_1^2 y - y_1^3 = -x_1^2 x + x_1^3$

$\therefore \quad y_1^2 y + x_1^2 x = a^3 \qquad x_1^3 + y_1^3 = a^3$

Now, $\quad y_1^2 y_2 + x_1^2 x_2 = y_1^3 + x_1^3 \qquad$ [Given]

$\therefore \quad y_1^2\left(y_2 - y_1\right) = x_1^2\left(x_1 - x_2\right)$

$x_2^3 + y_2^3 = a^3 \qquad$ By solving

We have,

$x_1^3 + y_1^3 = a^3$

$3x_1^2 x_2 + 3y_1^2 y_2 = 3a^3$

$3x_1 x_2^2 + 3y_1 y_2^2 = 3a^3$

$x_2^3 + y_2^3 = a^3$

$\therefore \quad (x_1+x_2)^3+(y_1+y_2)^3=(2a)^3$

8. Given $\frac{x^2}{a^2+k_1}+\frac{y^2}{b^2+k_1}=1$ ...(1)

and $\frac{x^2}{a^2+k_2}+\frac{y^2}{b^2+k_2}=1$ ...(2)

Subtracting (2) from (1), we get

$$x^2\left(\frac{1}{a^2+k_1}-\frac{1}{a^2+k_2}\right)+y^2\left(\frac{1}{b^2+k_1}-\frac{1}{b^2+k_2}\right)=0$$

$$\Rightarrow \quad x^2\left(\frac{k_2-k_1}{(a^2+k_1)(a^2+k_2)}\right)+y^2\left(\frac{k_2-k_1}{(b^2+k_1)(b^2+k_2)}\right)=0$$

$$\therefore \quad \frac{x^2}{y^2}=-\frac{(a^2+k_1)(a^2+k_2)}{(b^2+k_1)(b^2+k_2)} \qquad ...(3)$$

Now form (1), $\frac{2x}{(a^2+k_1)}+\frac{2y}{(b^2+k_1)}\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}=-\frac{x(b^2+k_1)}{y(a^2+k_1)}=m_1$ (say)

Similarly form (2), $\frac{dy}{dx}=-\frac{x(b^2+k_2)}{y(a^2+k_2)}=m_2$ (say)

$\therefore \quad m_1m_2=\frac{x^2(b^2+k_1)(b^2+k_2)}{y^2(a^2+k_1)(a^2+k_2)}$ {From (3)}

$=-1$

Hence given curves intersect orthogonally.

9. Let $P_1(t_1,t_1^3)$ is a point on the curve $y=x^3$

$\therefore \quad \left.\frac{dy}{dx}\right|_{(t_1,t_1^3)}=3t_1^2$

Tangent at P_1 is $y-t_1^3=3t_1^2(x-t_1)$ ...(1)

The intersection of (1) and $y=x^3$

$\Rightarrow \quad x^3-t_1^3=3t_1^2(x-t_1) \quad \Rightarrow \quad (x-t_1)(x^2+xt_1+t_1^2)-3t_1^2(x-t_1)=0$

$\Rightarrow \quad (x-t_1)^2(x+2t_1)=0$

If $P_2(t_2,t_2^3)$, then $(t_2-t_1)^2(t_2+2t_1)=0$

$\therefore \quad t_2=-2t_1 \; (t_2\neq t_1)$

Similarly, the tangent at P_2 will meet the curve at the point $P_3\left(t_3\ t_3^3\right)$ when $t_3=-2t_2=4t_1$ and so on.

The abscissae of $P_1, P_2, \ldots.., P_n$ are $t_1, -2t_1, 4t_1, \ldots, (-2)^{n-1} t_1$ in G.P.

$\therefore \quad \frac{t_2}{t_1}=\frac{t_3}{t_2}=\frac{t_4}{t_3}=\ldots=-2$ (r say)

$\therefore \quad t_2=t_1 r, t_3=t_2 r$ and $t_4=t_3 r$

$$\therefore \quad \text{Area of } \Delta P_1P_2P_3=\frac{1}{2}\begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix}$$

$$\text{and} \quad \text{Area of } \Delta P_2P_3P_4=\frac{1}{2}\begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} r t_1 & r^3 t_1^3 & 1 \\ r t_2 & r^3 t_2^3 & 1 \\ r t_3 & r^3 t_3^3 & 1 \end{vmatrix}=r^4 \text{ (Area of } \Delta P_1P_2P_3)$$

$$\therefore \quad \frac{\text{Area of}\left(\Delta P_1P_2P_3\right)}{\text{Area of}\left(\Delta P_2P_3P_4\right)}=\frac{1}{r^4}=\frac{1}{(-2)^4}=\frac{1}{16}.$$

10. Given curve is $x^3+y^3=c^3$

$\therefore \quad \frac{dy}{dx}=-\frac{x^2}{y^2} \qquad \Rightarrow \qquad \left.\frac{dy}{dx}\right|_{(a,b)}=-\frac{a^2}{b^2}$ ...(1)

and tangent at (a,b) cuts the curve again at $\left(a_1, b_1\right)$

$\therefore \quad$ Slope of tangent $=\frac{b_1-b}{a_1-a}=-\frac{a^2}{b^2}$ {From (1)} ...(1)

Also $\quad a^3+n^3=c^2$ ...(2)

and $\quad a_1^3+b_1^3=c^3$ ...(3)

subtracting (3) from (2), we get $\left(a^3-a_1^3\right)+\left(b^3-b_1^3\right)=0$

$\Rightarrow \quad \left(a-a_1\right)\left(a^2+a a_1+a_1^2\right)+\left(b-b_1\right)\left(b^2+b b_1+b_1^2\right)=0$

$\Rightarrow \quad \frac{b_1-b}{a_1-a}=-\frac{a^2+aa_1+a_1^2}{b^2+bb_1+b_1^2}$ ...(4)

From (1) and (4), $\quad -\frac{a^2}{b^2}=-\frac{a^2+aa_1+a_1^2}{b^2+bb_1+b_1^2}$

$\Rightarrow \quad a^2b^2+a^2bb_1+a^2b_1^2=a^2b^2+ab^2a_1+a_1^2b^2$

$\Rightarrow \quad ab\left(ab_1-ba_1\right)+a^2b_1^2+a_1^2b^2=0$

$\Rightarrow \quad ab\left(ab_1-ba_1\right)+\left(ab_1+a_1b\right)\left(ab_1-a_1b\right)=0$

$\Rightarrow \quad \left(ab_1-a_1b\right)+\left(ab+ab_1+a_1b\right)=0$

If $ab_1 - a_1 b = 0$

then $\frac{a}{a_1} = \frac{b}{b_1} = \frac{\sqrt[3]{a^3 + b^3}}{\sqrt[3]{a_1^3 + b_1^3}} = \frac{c}{c} = 1$ (Law of proportion)

$\therefore$ $a = a_1$ and $b = b_1$ which is impossible.

Hence $ab + ab_1 + a_1 b = 0$ or $\frac{a_1}{a} + \frac{b_1}{b} = -1$

11. $|\sin x| + |\cos x| \in \left[1, \sqrt{2}\right]$

$\therefore$ $\left[|\sin x| + |\cos x|\right] = 1$

For $x^2 + y^2 = 5, \frac{dy}{dx} = -\frac{x}{y}$

For $y = 1, x = \pm 2$

$\therefore$ $m_1 = 0, m_2 = \mp 2$

$\therefore$ $\tan\alpha = \left|\frac{\pm 2 - 0}{1 + 0}\right| = 2$

$\therefore$ $\alpha = \tan^{-1}(2)$

12. $x^m y^n = k^{m+n}$

$\therefore$ $\frac{dy}{dx} = -\frac{my}{nx}$

$\therefore$ $\frac{Y - y}{X - x} = -\frac{my}{nx}$

$\therefore$ $nxY - nxy = -myX + mxy$

$\therefore$ $myX + nxY = xy(m + n)$

$X = 0 \quad \Rightarrow \quad Y = \frac{y(m+n)}{n}$

$Y = 0 \quad \Rightarrow \quad X = \frac{x(m+n)}{m}$

Now, $x = \frac{\frac{Rx(m+n)}{m} + 0}{k + 1}$, Where R is ratio in which point of contact divides the segments

$\therefore$ $mR + m = Rm + Rn$

$\therefore$ $R = \frac{m}{n}$

$\therefore$ Ratio $= m : n$

13. $\frac{x^2}{a}+\frac{y^2}{b}=1 \ : \frac{x^2}{a_1}+\frac{y^2}{b_1}=1$

$m_1 = -\frac{bx}{ay}$, $m_2 = -\frac{b_1x}{a_1y}$

solving, $x^2 = a, y^2 = b_1$

$\therefore \quad m_m m_2 = \frac{bb_1a}{aa_1b_1} = \frac{b}{a_1} = -1$

$\therefore$ Condition for orthogonality is $a_1 + b = 0$.

14. T.P. $\frac{\sin\theta}{\theta} < \frac{\sin(\sin\theta)}{\sin\theta}$

Let $f(x) = \frac{\sin x}{x}$

$\therefore \quad f'(x) = \frac{x\cos x - \sin x}{x^2}$

Let $g(x) = x\cos x - \sin x$

$\therefore \quad g'(x) = -x\sin x < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$g(0) = 0$

$\therefore \quad g(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad f'(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad f(x)$ is M.D. in $\left(0, \frac{\pi}{2}\right)$

Now $\theta > \sin\theta$ for $x \in \left(0, \frac{\pi}{2}\right)$ (as shown in Q.9)

$\therefore \quad f(\theta) < f(\sin\theta)$ Hence provrd.

15. $f(x) = \frac{1-x+x^2}{1+x+x^2}$

$\therefore \quad f'(x) = \frac{(2x-1)(x^2+x+1)-(2x+1)(x^2-x+1)}{(x^2+x+1)^2}$

$$= \frac{(2x^3+x^2+x-1)-(2x^3-x^2+x+1)}{(x^2+x+1)^2}$$

$$= \frac{2(x+1)(x-1)}{(x^2+x+1)^2}$$

Now, $x^2 + x + 1 > 0 \quad \forall\, x \in R$

$\therefore$ $f(x)$ is MI in $(-\infty,-1)\cup(1,\infty)$ and MD in $(-1,1)$.

16. $f(x) = (m+2)x^3 - 3mx^2 + 9mx - 1$

$\therefore$ $f'(x) = 3(m+2)x^2 - 6mx + 9m < 0 \quad \forall\, x \in R$

$\therefore$ $(m+2)x^2 - 2mx + 3m < 0 \quad \forall\, x \in R$

$\therefore$ $m + 2 < 0 \quad \Rightarrow \quad m < -2$

and $4m^2 - 4(m+2)(3m) < 0$

$\therefore$ $m^2 - (m+2)(3m) < 0$

$\therefore$ $-2m^2 - 6m < 0$

$\therefore$ $m(m+3) > 0$

$\therefore$ $m < -3$ as $m < -2$

$\therefore$ $m \in (-\infty, -3)$

17. $\frac{\sin x}{x}$ is MD is proved in Q.10

Now $\lim_{x\to 0} \frac{\sin x}{x} = 1$ & $\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$

$\therefore$ $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ $\frac{2x}{\pi} < \sin x < x$ for $x \in \left(0, \frac{\pi}{2}\right)$

18. Refer assignment Q.10

19. $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$

$\therefore$ $h'(x) = f'(x) - 2f(x)[f'(x)] + 3[f(x)]^2 f'(x) > 0$

$\therefore$ $f'(x)\underbrace{\left(3f^2(x) - 2f(x) + 1\right)}_{>0 \text{ for all } f(x)\in R} > 0$

$\therefore$ $f'(x) > 0$ for $h'(x) > 0$

20. $f(x) = \begin{cases} xe^{ax} & , \quad x \le 0 \\ x + ax^2 - x^3 & , \quad x > 0 \end{cases}$, $a > 0$

$f'(x) = e^{ax}(1 + ax), x \le 0$

$1 + 2ax - 3x^2, x > 0$

$f'(x) > 0$

For $x \le 0$

$f'(x) > 0$ for $x \in \left(\frac{1}{a}, 0\right]$

For $x > 0$,

$3x^2 - 2ax - 1 < 0$ for $f'(x) > 0$

$\therefore \quad x \in \left[0, \frac{a + \sqrt{a^2 + 3}}{3}\right)$

$\therefore \quad f(x)$ is monotonically increasing in $x \in \left(\frac{1}{a}, \frac{a + \sqrt{a^2 + 3}}{3}\right)$

21. $f''(x) > 0 \quad \forall x \in R$

$Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$\therefore \quad Q(x) = 2x f'\left(\frac{x^2}{2}\right) - 2xf'(6 - x^2) > 0$

$\therefore \quad 2x\left(f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right) > 0$

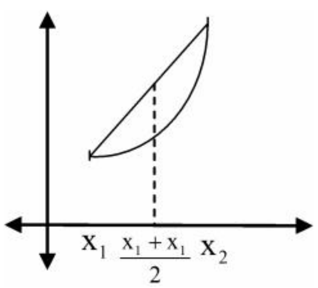
$\therefore \quad x\left(f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right) > 0$

$\frac{x^2}{2} = 6 - x^2 \quad \Rightarrow \quad x = \pm 2$

$\therefore \quad Q(x)$ is MI in $(-2, 0) \cup (2, \infty)$ & MD in $(-\infty, -2) \cup (0, 2)$

22. $f'(x) > 0$, $f''(x) > 0$, $x_1 < x_2$

$\therefore$ Graph of f(x) is



By graph

$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$

23. $Q(x) = 2f\left(\frac{x}{2}\right) + f(2-x)$

$\therefore \quad Q'(x) = f'\left(\frac{x}{2}\right) - f'(2-x)$

$f''(x) < 0$

$\therefore \quad g'(x) > 0$

$\Rightarrow \quad f'\left(\frac{x}{2}\right) > f'(2-x)$

$\Rightarrow \quad \frac{x}{2} < 2 - x$

$\therefore \quad x < \frac{4}{3}$

$\therefore \quad Q(x)$ increases in $\left(-\infty, \frac{4}{3}\right)$ & decreases in $\left(\frac{4}{3}, \infty\right)$

24. $f(x) = \sin^3 x + \lambda \sin^2 x$

$\therefore \quad f'(x) = 3\sin^2 x \cos x + 2\lambda \sin x \cos x = 0$

$\therefore \quad \cos x\left(3\sin^2 x + 2\lambda \sin x\right) = 0$

$\therefore \quad \cos x \neq 0$ as $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad \sin x\left(3\sin x + 2\lambda\right) = 0$

$\therefore \quad 3\sin x + 2\lambda = 0$ has a root

$\therefore \quad \sin x = -\frac{2\lambda}{3}$ has a root

$\therefore \quad \lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$ as $\sin x = 0$ is already a root.

25. $f(x)$ is 6^{th} degree polynomial

$$\lim_{x\to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{\frac{1}{x}} = r = e^2$$

$\therefore \quad e^{\lim_{x\to 0}\frac{f(x)}{x^4}} = e^2 \qquad f(x) = ax^6 + bx^5 + cx^4$ & $c = 2$

$\therefore \quad$ Roots are 1 & 2

$2 = \frac{8}{6a} \qquad \therefore \qquad a = \frac{2}{3}$

$\therefore \quad f'(x) = 6ax^5 + 5bx^4 + 8x^3 = 0$

$\therefore \quad 6ax^2 + 5bx + 8 = 0$ has roots 1 & 2

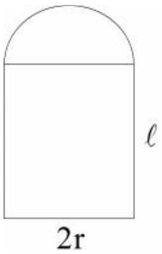
$\therefore \quad -\frac{5b}{6a} = 3$ and $\frac{8}{6a} = 2$

$\therefore \quad a = \frac{2}{3} \qquad b = -3 \times 6 \times \frac{2}{3} \times \frac{1}{5}$

$b = -\frac{12}{5}$

$\therefore \quad f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$

26.



$P = \pi r + 2r + 2\ell + 2r$

$= 4r + 2\ell + \pi r$

$\therefore \quad \frac{dP}{dr} = 4 + \pi + \frac{2d\ell}{dr} = 0$

$\frac{d\ell}{dr} = -\frac{(4+\pi)}{2}$

$L = 3 \times 2r\ell + \frac{\pi r^2}{2}$

$\therefore \quad \frac{dL}{dr} = 0$

$\therefore \quad 6\ell + 6r\left(\frac{4+\pi}{2}\right) + \pi r = 0$

$\therefore \quad 3\ell = 12r + 2\pi r$

$\frac{\ell}{r} = \frac{12 + 2\pi}{3}$

27.

(a) $\frac{dx}{dt} = 6t^2 - 6t + 6$

$\frac{dy}{dt} = 6t^2 + 6t + 6$

$\therefore \quad \frac{dy}{dx} = \frac{6t^2 + 6t + 6}{6t^2 - 6t + 6}$

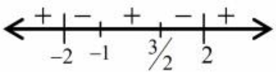
$= \frac{t^2 + t + 1}{t^2 - t + 1}$

Solving it is maximum at $t = 1$ and has max value = 3.

$\therefore \quad$ Point on curve $= (5, 11)$

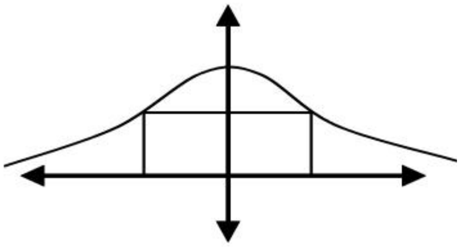
(b) $\frac{dy}{dx} = \frac{12t^2 - 6t - 18}{5t^4 - 15t^2 - 20} = \frac{6}{5}\frac{\left(2t^2 - t - 3\right)}{\left(t^4 - 3t^2 - 4\right)}$

$= \frac{6(2t-3)(t+1)}{5(t+2)(t-2)}$



$\therefore$ $f(x)$ has minimum value at $t = -1$ and max at $t = \frac{3}{2}$

28.



By symmetry, points on x-axis will be $(-a, 0)$ and $(a, 0)$

$A = 2ae^{-a^2}$

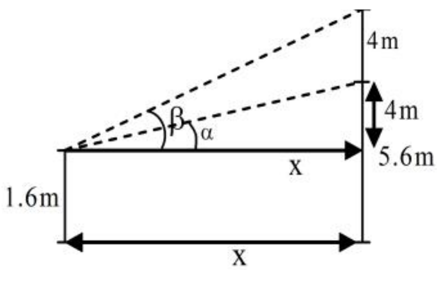
$\therefore$ $\frac{dA}{da} = 2e^{-a^2} - (2a)^2 e^{-a^2} = 0$

$\therefore$ $2a^2 = 1$

$\therefore$ $a = \pm\frac{1}{\sqrt{2}}$

$\therefore$ $A = \sqrt{2}\, e^{-\frac{1}{2}} = \sqrt{\frac{2}{e}}$

29.



$\tan\alpha = \frac{4}{x}, \tan\beta = \frac{8}{x}$

$\therefore$ $\beta - \alpha = \tan^{-1}\left(\frac{8}{x}\right) - \tan^{-1}\left(\frac{4}{x}\right) = \tan^{-1}\left(\frac{\frac{4}{x}}{1 + \frac{32}{x^2}}\right) = \tan^{-1}\left(\frac{4x}{x^2 + 32}\right)$

$\tan^{-1} x$ is an increasing function

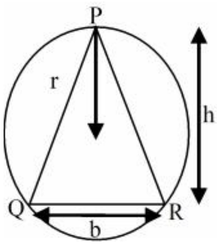
$\therefore \quad \text{max of } \dfrac{4x}{x^2+32}$

$\Rightarrow \quad \text{max of } \tan^{-1}\left(\dfrac{4x}{x^2+32}\right)$

$f(x) = \dfrac{4x}{x^2+32}$

$\therefore \quad f'(x) = \dfrac{4(x^2+32) - 8x^2}{(x^2+32)^2} = 0$

$\therefore \quad 128 = 4x^2 \qquad \therefore \quad x = \sqrt{32}\,m = 4\sqrt{2}$

31. 

$$(h-r)^2 + \frac{b^2}{4} = r^2$$

Solving as in (level 1) Q.12 We get the answer.

32. Q is (1,0)

Let equation of circle wilth centre be

$(x-1)^2 + y^2 = r^2$

$x^2 + y^2 = 1$

$\therefore \quad -2x + 1 = r^2 - 1$

$\therefore \quad x = \dfrac{2-r^2}{2}$

$\therefore \quad y^2 = 1 - \left(\dfrac{2-r^2}{2}\right)^2 = 1 - 1 + r^2 - \dfrac{r^4}{4}$

$\therefore \quad y = r\sqrt{-\dfrac{r^2}{4} + 1}$

$\therefore \quad \Delta = \dfrac{1}{4} r^2 \sqrt{4-r^2}$

$\therefore \quad \dfrac{d\Delta}{dr} = \dfrac{r}{2}\sqrt{4-r^2} - \dfrac{r^2}{4} \times \dfrac{r}{\sqrt{4-r^2}} = 0$

$\therefore \quad \dfrac{r}{2}\sqrt{4-r^2} = \dfrac{r^3}{4\sqrt{4-r^2}}$

$\therefore \quad 2r(4-r^2) = r^3$

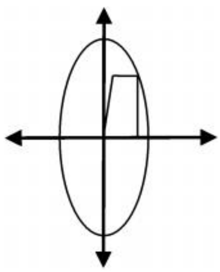
$\therefore \quad r = 0 \quad \text{or} \quad 8 - 2r^2 = r^2$

$\therefore \quad r = \sqrt{\frac{8}{3}}$

$\therefore \quad \sqrt{4 - r^2} = \frac{2}{\sqrt{3}}$

$\therefore \quad \Delta = \frac{1}{4} \times \frac{8}{3} \times \frac{2}{\sqrt{3}} = \frac{4}{3\sqrt{3}}$

33.



$\Delta = \frac{1}{2} \times \beta \times (\alpha + \alpha - \beta)$

$= \frac{1}{2}\beta(2\alpha - \beta)$

$= \alpha\beta - \frac{\beta^2}{2}$

$4\alpha^2 + 3\beta^2 = 12$

$\therefore \quad 8\alpha\frac{d\alpha}{d\beta} + 6\beta = 0$

$\therefore \quad \frac{d\alpha}{d\beta} = -\frac{3\beta}{4\alpha}$

$\therefore \quad \frac{d\Delta}{d\beta} = \alpha - \beta\left(\frac{3\beta}{4\alpha}\right) - \beta = 0 \qquad \therefore \quad 4\alpha^2 - 4\alpha\beta - 3\beta^2 = 0$

$\therefore \quad (2\alpha + \beta)(2\alpha - 3\beta) = 0 \qquad \text{But } \alpha\beta > 0$

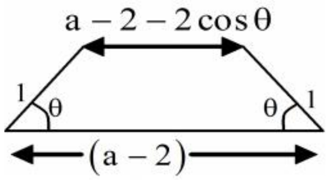
$\therefore \quad 2\alpha = 3\beta \qquad \therefore \quad 4\alpha^2 = 9\beta^2$

$4\alpha^2 + 3\beta^2 = 12$

$\therefore \quad \beta = 1 \text{ and } \alpha = \frac{3}{2} \qquad ....(\alpha, \beta > 0)$

Point is $\left(\frac{3}{2}, 1\right)$

34.



$h = \sin\theta$

$\therefore \quad A = \frac{1}{2}\sin\theta(2a - 4 - 2\cos\theta)$

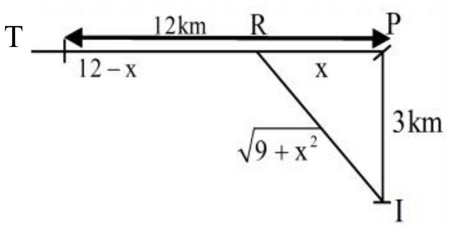
$= a\sin\theta - 2\sin\theta - \sin\theta\cos\theta$

$\therefore \quad \frac{dA}{d\theta} = a\cos\theta - 2\cos\theta - \cos^2\theta + \sin^2\theta = 0$

$\therefore \quad 2\cos^2\theta + (2 - a)\cos\theta - 1 = 0 \qquad \theta \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \cos\theta = \frac{a - 2 + \sqrt{(a-2)^2 + 8}}{4}$

35.



$\therefore$ Rate of cable $= 4000\sqrt{94x^2} + 2000(12 - x)$

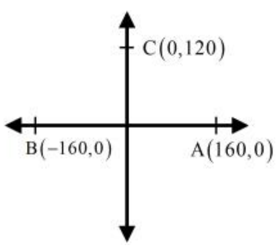
$\therefore \quad \frac{dR}{dx} = \frac{4000x}{\sqrt{9 + x^2}} - 2000 = 0$

$\therefore \quad 2x = \sqrt{9 + x^2}$

$\therefore \quad 4x^2 = x^2 + 9$

$\therefore \quad x = \sqrt{3}$ km from P towards T.

36. Solving we get $A \equiv (160, 0), B \equiv (-160, 0), C \equiv (0, 120)$



As ΔABC is an isosceles Δ, by symmetry, godown should be located on u-axis.

Let godown be lovateda at (0,y) , $0 < y < 120$

$\therefore$ Sum of distance from godown is

$s = 120 - y + 2\sqrt{y^2 + 25600}$

$\therefore \quad \frac{ds}{dy} = -1 + \frac{2y}{\sqrt{y^2 + 25600}}$

$\therefore \quad 4y^2 = y^2 + 25600$

$\therefore \quad y^2 = \frac{25600}{3}$

$\therefore \quad y = \frac{160}{\sqrt{3}}$

$\therefore$ Godown should be located at $\left(0, \frac{160}{\sqrt{3}}\right)$.

37. $P = (1200 - x)(725 + x)$

$\therefore \quad \frac{dP}{dx} = -725 - x + 1200 - x = 0$

$\therefore \quad x = \frac{475}{2} = 237.5$

But as number of subscribers is an integer $x = 237$ or 238 (Values are same)

$\therefore$ Subscribers $= 962$ or 963.

38. $g(t) = 3t^4 - 8t^3 - 6t^2 + 24t$

$g'(t) = 12t^3 - 24t^2 - 12t + 24 = 0$

$\therefore \quad t^3 - 2t^2 - t + 2 = 0$

$\therefore \quad (t^2 - 1)(t - 2) = 0$

By examining it has maxima at $t = 1$ and minima at $t = 2$

$\therefore \quad f(x) = 3x^4 - 8x^3 - 6x^2 + 24x \ ; \ 1 \le x < 2$

$h(t) = 3t + \frac{1}{4}\sin^2 \pi t + 2$

$\therefore \quad h'(t) = 3 + \frac{\pi}{4} \times \sin 2\pi t > 0 \qquad \forall\, t$

$\therefore \quad f(x)$ has minima at $x = 2$

$\therefore \quad f(x) = 3x^4 - 8x^3 - 6x^2 + 24x \ ; \ 1 \le x < 2$

$\quad = 3x + \frac{1}{4}\sin^2 \pi x + 2 \ ; \ 2 \le x \le 4$

$f(1) = 3 - 8 - 6 + 24 = 13$

$\lim_{x \to 2^-} f(x) = 48 - 64 - 24 + 48 = 8$

$f(2) = 6 + 2 = 8$

$f(4) = 12 + 2 = 14$

$\therefore$ Greatest value of $f(x)=14$ and least value of $f(x)=8$.

40. Refer Ex. level 1 (Q.21) for method

We get $\dfrac{b^3-b^2+b-1}{b^2+3b+2}\ge 0$

$\therefore \quad \dfrac{(b^2+1)(b-1)}{(b+1)(b+2)}\ge 0$

$\therefore \quad b\in(-2,-1)\cup[1,\infty)$.

41. $f(x)=x+\dfrac{a}{x^2}-2>0 \qquad \forall\, x\in R$

$\therefore \quad f'(x)=1-\dfrac{2a}{x^3}=0$

$\Rightarrow \quad x=(2a)^{\frac{1}{3}}$

$\therefore \quad (2a)^{\frac{1}{3}}+\dfrac{a^{\frac{1}{3}}}{2^{\frac{1}{3}}}-2>0$

$\therefore \quad \left(4^{\frac{1}{3}}+1\right)a^{\frac{1}{3}}>2^{\frac{4}{3}}$

$\therefore \quad a>\dfrac{2^4}{\left(4^{\frac{1}{3}}+1\right)^3}$

$\left(4^{\frac{1}{3}}+1\right)^3=5+3\left(4^{\frac{2}{3}}+4^{\frac{1}{3}}\right)$

$\therefore \quad a>1$ ....

$\therefore$ Least natural number is 2.

42. $f'(c)=2c-2=-1$

$\therefore \quad c=\dfrac{1}{2}$

$\therefore \quad x=\dfrac{1}{2},\ y=\dfrac{1}{4}-1+3=\dfrac{9}{4}$

$\therefore \quad y-\dfrac{9}{4}=\dfrac{1}{2}-x$

$\therefore \quad x+y=\dfrac{11}{4}$

i.e. $\quad 4x+4y=11$

43. $\dfrac{f'(c)}{g'(c)} = \dfrac{f(1)-f(0)}{g(1)-g(0)} = \dfrac{4}{2} = 2$

44. $f(x) = x + e^x$

$\therefore \quad f'(x) = 1 + e^x > 0 \quad \forall\, x \in R$

Now, $f(-\infty) = -\infty,\ f(\infty) = \infty$

$\therefore \quad f(x)$ has only one real root.

45. $f(0) = 4\ ,\ f(5) = -1$

$g(x) = \dfrac{f(x)}{x-1}$

$g(0) = -4\ ,\ g(5) = -\dfrac{1}{4}$

$\therefore \quad g'(c) = \dfrac{4 - \frac{1}{4}}{5} = \dfrac{3}{4}$

46. $f(x)$ and $f'(x)$ should have a common root

$f(x) = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\cdots + \dfrac{x^n}{n!}$

$\therefore \quad f'(x) = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots\cdots + \dfrac{x^{n-1}}{(n-1)!}$

Which do not have a common root.

47. $f(x) = x + \cos x - a$

$f'(x) = 1 - \sin x \geq 0 \quad \forall\, x \in R$

$\therefore \quad f(x)$ is an increasing function

If $a < 1\ ,\ f(0) > 0$

$\therefore$ No positive root

If $a > 1\ ,\ f(0) < 0$

$\therefore$ one positive root.

48. Consider the function $f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx + d$

We have $f(0) = d$ and

$f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c + d = \dfrac{2a + 3b + 6c}{6} + d = 0 + d = d \qquad (\because\ 2a + 3b + 6c = 0)$

Therefore, 0 and 1 are the roots of the polynomial f(x).

Consequently, there exists at least one root of the polynomial $f'(x) = ax^2 + bx + c$ lying between 0 and 1.