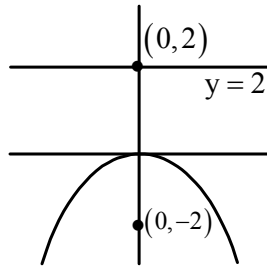


PARABOLA BOOKLET SOLUTION
EXERCISE – 1 (A)

1. (B)



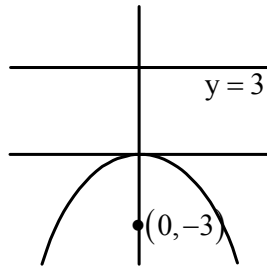
$$x^2 = -8y$$

$$x^2 = -4Ay$$

$$\Rightarrow A = 2$$

Directrix is $y = 2$

2. (A)



$$x^2 = -12y$$

3. (C)

Vertex $(0, 0)$

Directrix $3x - 4y + 2 = 0$

$$D = \frac{|2|}{\sqrt{3^2 + 4^2}} = \frac{2}{5} = a$$

$$L.R = 4a = 4 \cdot \frac{2}{5} = \frac{8}{5}$$

4. (B)

Focus of $y^2 = 8x$ is $(-2, 0)$

Focal chord $2x + y + d = 0$

$$\Rightarrow 2(-2) + 0 + \lambda = 0$$

$$\Rightarrow \lambda = 4$$

5. (A)

Focal distance is $x + a$

$$\therefore x, +3$$

6. (B)

Vertex is (a, b)

$$L.R = \ell = 4A \Rightarrow A = \ell/4$$

$$(x - a)^2 = \ell(y - b)$$

7. (A)

$$x^2 = -y, \quad x^2 = -4AV$$

$$\Rightarrow 4A = 1 = \ell$$

8. (C)

$D = 2a$ between focus & Directrix

$$x^2 = -8y$$

$$\Rightarrow 2a = 4$$

9. (B)

$$F(3,0) \quad \ell = 4A = 8 \Rightarrow A = 2$$

$$\Rightarrow \text{vertex } (1,0)$$

10. (B)

$$F(2,1)$$

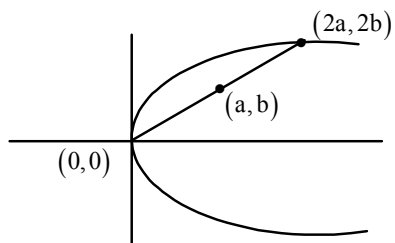
Let the L. R. be $2x - 3y + c = 0$

$$\text{Now, } 2(2) - 3(1) + c = 0$$

$$\Rightarrow c = -1$$

$$LR \equiv 2x - 3y - 1 = 0$$

11. (D)



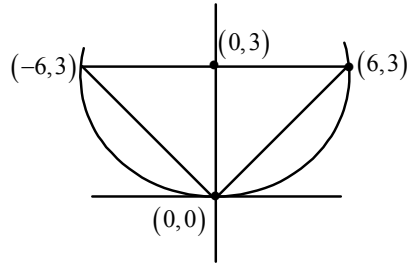
$$y^2 = 4x$$

$$(2b)^2 = 4 \cdot 2a$$

$$4b^2 = 8a$$

$$\Rightarrow b^2 = 2a$$

12. (C)



$$A = \frac{1}{2} \times 12 \times 3$$

$$A = 18 \text{ sq. units}$$

13. (A)

$$9x^2 - 6x + 36y + 9 = 0$$

$$(3x)^2 - 2 \cdot 3x \cdot 1 + 1 + 36y + 8 = 0$$

$$(3x - 1)^2 = -36y - 8$$

$$9\left(x - \frac{1}{3}\right)^2 = -36\left(y + \frac{2}{9}\right)$$

$$\text{Vertex } \left(\frac{1}{3}, -\frac{2}{9}\right)$$

14. (C)

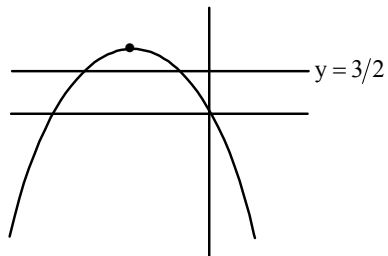
$$x^2 + 4x + 2y = 0$$

$$(x + 2)^2 = -2y + 4$$

$$(x + 2)^2 = -2(y - 2)$$

$$\text{Now, } 4a = 2$$

$$\Rightarrow a = \frac{1}{2}$$



$$\text{LR} \equiv y = \frac{3}{2}$$

15. (D)

$$y^2 = x - 2y + 2 = 0$$

$$(y - 1)^2 = x - 1$$

$$\text{Vertex } (1, 1), a = \frac{1}{4}$$

$$\text{Focus } \Rightarrow \left(\frac{5}{4}, 1\right)$$

16. (D)
 $y^2 + 2y + x = 0$
 $(y+1)^2 = -(x-1)$
 Vertex $(1, -1)$
17. (C)
 $x^2 - 4x - 3y + 10 = 0$
 $(x-2)^2 = 3(y-2)$
 Axis is $x-2=0$
18. (D)
 $x^2 + 4x + 2y - 7 = 0$
 $(x+2)^2 = -2(y-11/2)$
 Vertex $(-2, 11/2)$
19. (B)
 $y^2 = 4y - 2x - 8 = 0$
 $(y-2)^2 = 2(x+6)$
 LR = 2
20. (A)
 $y^2 + 4x + 2y - 8 = 0$
 $(y+1)^2 = -4(x-9/4)$
 It is focus $(5/4, -1)$
21. (C)
 $x = ay^2 + by + c$
 $x = a \left[(y + b/2a)^2 - \frac{b^2}{4a^2} + c/a \right]$
 $\frac{1}{a} \left(x + \frac{b^2 - 4ac}{4a} \right) = (y + b/2a)^2$
 L.L.R = $1/a$
22. (C)
 For parabola $4^2 = ab$
 Option (C) $(4) \neq 2 \times 1$
 $2x^2 + y^2 - 4xy = 8$ is not a parabola

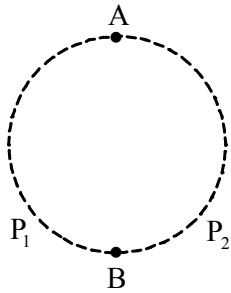
23. (B)
Option (B)

$$\frac{x}{y} + \frac{2y}{x} = 0$$

$$x^2 = -24y$$

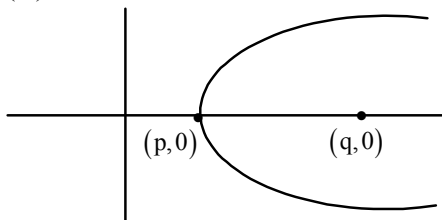
24. (C)
Theory based

25. (C)
Given ends A & B two parabolas are possible.



26. $v(0, a)$
 $f(0, 0)$
 $(x - 0)^2 = -4a(y - a)$
 $x^2 = 4a(a - y)$

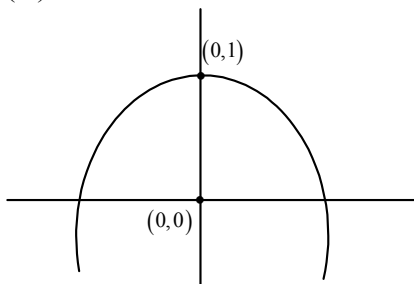
27. (C)



$$y^2 = 4(q - p)(x - p)$$

$$\Rightarrow y^2 = -4(p - q)(x - p)$$

28. (A)



$$(x-0)^2 = -4 \times 1(y-1)$$

$$x^2 + 4y - 4 = 0$$

29. (B)

$$v(2,0) \quad f(5,0)$$

$$y^2 = 4 \times 3(x-2)$$

$$y^2 = 12x - 24$$

30. (D)

$$x-2 = t^2, y = 2t$$

$$x-2 = (y/2)^2$$

$$\Rightarrow y^2 = 4(x-2)$$

31. (B)

$$x = \frac{t}{4}, y = \frac{t^2}{4}$$

$$\Rightarrow y = \frac{(4x)^2}{4} = 4x^2 \text{ a parabola}$$

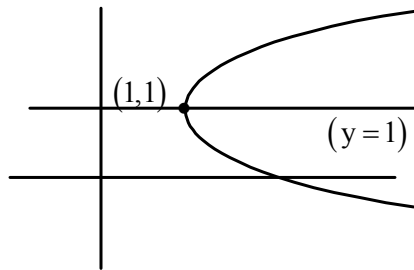
32. (A)

$$x = t^2 + 1, y = 2t + 1$$

$$x = \left(\frac{y-1}{2}\right)^2 + 1$$

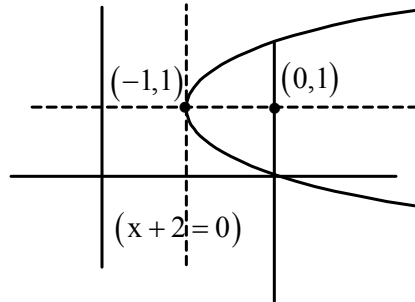
$$4x = (y-1)^2 + 4$$

$$\Rightarrow (y-1)^2 = 4(x-1)$$



Directrix is $x = 0$

33. (D)



Vertex $(-1, 1)$

$f(0, 1)$

$$(y - 1)^2 = 4 \times 1(x + 1)$$

$$\text{Now, } x + 1 = 1 \times t^2 \Rightarrow x = t^2 - 1$$

$$y - 1 = 2 \times 1 \times t \Rightarrow y = 2t + 1$$

Any point $(t^2 - 1, 2t + 1)$

34. (A)

$(a/m^2, 2/m)$ doesn't satisfy $y^2 = 4ax$

35. (A)

$$y^2 - 12x - 2y - 11 = 0$$

$$(y - 1)^2 = 12(x + 1)$$

$$y - 1 = 2 \times 3 \times t \quad x + 1 = 3.t^2$$

$$y = 1 + 6t \quad x = 3t^2 - 1$$

36. (B)

$$y^2 = 6x$$

Vertex $(0, 0)$

Negative end of LR $(3/2, -3)$

$$\frac{y - 0}{x - 0} = \frac{3 - 0}{3/2 - 0} \Rightarrow \frac{3}{2}y = -3x$$

$$\Rightarrow y + 2x = 0$$

37. (A)

$f(8, 0)$

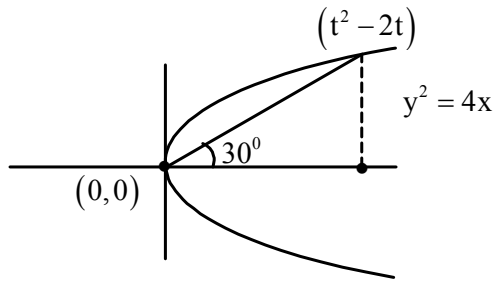
One end $(2, -8)$

$$\frac{-2a}{t} = -8 = \frac{-2 \times 8}{t} \Rightarrow t = 2$$

$$(at^2, 2at) = (8 \times 2^2, 2 \times 8 \times 2)$$

$$\equiv (32, 32)$$

38. (C)



$$\tan 30^\circ = \frac{2t}{t^2}$$

$$\frac{1}{\sqrt{3}} = \frac{2}{t} \Rightarrow$$

$$t = 2\sqrt{3}$$

$$\begin{aligned} \text{Length} &= \sqrt{\left((2\sqrt{3})^2\right)^2 + (2 \times 2\sqrt{3})^2} \\ &= \sqrt{12^2 + 4^2 \times 3} \\ &= 8\sqrt{3} \end{aligned}$$

39. (B)

$$y^2 = 4x$$

$$(-x)^2 = 4x$$

$$x^2 = 4x \Rightarrow x = 0 \text{ or } x = 4$$

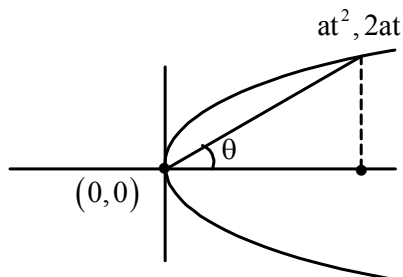
$$y^2 = 0 \quad \text{or} \quad y^2 = 4 \times 4$$

$$4 = 0 \quad \text{or} \quad y = \pm 4$$

$$A \equiv (4, 4) \quad B \equiv (4, -4)$$

$$AB = 8$$

40. (B)



$$\tan \theta = \frac{2at}{at^2}$$

$$t = 2 \cot \theta$$

$$\text{Length} = \sqrt{a^2 t^4 + 4a^2 t^2}$$

$$\begin{aligned}
 &= at\sqrt{t^2+4} \\
 &= a \times 2 \cot \theta \sqrt{4 \cot^2 \theta + 4} \\
 &= a \times 2 \cot \theta \times 2 \times \cos \sec \theta \\
 &= 4a \cos \theta \cdot \cos \sec^2 \theta
 \end{aligned}$$

41. (C)

Put $y = 0$

$$x^2 = 7x + 12 = 0$$

$$\Rightarrow x = 4 \text{ or } 3$$

Intercept is 1 unit

42. (B)

Put $x = 0$

$$2y^2 + 6y - 8 = 0$$

$$y^2 + 3y - 4 = 0$$

$$(y + 3/2)^2 - 9/4 - 4 = 0$$

$$(y + 3/2)^2 = \frac{25}{4}$$

$$y + 3/2 = \pm 5/2$$

$$y = 1, -4$$

Intercept length = 5

43. (A)

$$x^2 + 3x = 5 - x$$

$$x^2 + 4x + 4 = 9$$

$$(x + 2)^2 = 9$$

$$x + 2 = \pm 3$$

$$x = 1, -5$$

$$y = 5 - x$$

$$= 5 - 1 \text{ or } 5 - (-5)$$

$$y = 4, 10$$

Points are $(1, 4)$ & $(-5, 10)$

$$\text{Length} = 6\sqrt{2}$$

44. (C)

$$2a = 2 \times 2 = \frac{2 \times sp \times sa}{sp + sq}$$

$$2 = \frac{6 \times s\theta}{6 + s\theta}$$

$$12 + 2s\theta = 65\theta$$

$$4s\theta = 12$$

$$5\theta = 3$$

45. (B)

$$2a = \frac{sp \times s\theta}{sp + s\theta} \quad \{A = 2a \text{ in this case}\}$$

$$2a = \frac{2 \times 4}{6} = 4/3$$

$$a = 2/3$$

46. (B)

$$y^2 = K(1-y)$$

$$y^2 + Ky - K = 0$$

$$D = 0 \Rightarrow K^2 = (4 \times -K) = 0$$

$$K^2 + 4K = 0$$

$$K = 0, -4$$

47. (C)

$$K - x = x - x^2$$

$$x^2 - 2x + K = 0$$

$$D = 0 \Rightarrow 4 - 4K = 0$$

$$\Rightarrow K = 1$$

48. (D)

$$y^2 = 4a(x+a)$$

$$y = mx + c$$

$$y = m(x+a) + c - am$$

$$\text{Now, } c - am = \frac{a}{m}$$

$$\Rightarrow c = am + \frac{a}{m}$$

49. (A)

$$(1-2x)^2 = 4x$$

$$1 + 4x^2 - 4x = 4x$$

$$4x^2 - 8x + 1 = 0$$

$$\therefore D > 0, \quad 2 \text{ real points}$$

50. (C)

$$y^2 = 4ax$$

$$\ell x + my + n = 0$$

$$y = -\frac{\ell x}{m} - \frac{n}{m}$$

$$c = \frac{a}{M} \Rightarrow \frac{-n}{m} = \frac{a}{-\ell/m}$$

$$\Rightarrow \frac{\ell n}{m^2} = a$$

51. (B)

$$x^2 + 4\left(\frac{x-k}{2}\right) = 0$$

$$x^2 + 2x - 2k = 0$$

$$D = 0 \Rightarrow 4 + 8K = 0$$

$$\Rightarrow K = -1/2$$

52. (D)

$$\left(my + \frac{a}{m}\right)^2 = 4ay$$

$$m^2 y^2 + \frac{a^2}{m^2} + 2ay = 4ay$$

$$(my - a/m)^2 = 0$$

$$y = \frac{a}{m^2}$$

$$\Rightarrow x = \frac{2a}{m}$$

$$\left(\frac{2a}{m}, \frac{a}{m^2}\right)$$

53. (A)

$$y^2 = \frac{1}{4}x$$

$$y = \sqrt{3}x + C$$

$$\text{Hence } c = a/m \Rightarrow c = \frac{1/16}{\sqrt{3}} = \frac{1}{16\sqrt{3}}$$

$$(x.y.) \equiv \left(\frac{1}{48}, \frac{1}{8\sqrt{3}}\right)$$

54. (A)

$$y.2 = 4 \cdot \frac{(x+1)}{2}$$

$$y = x + 1$$

55. (B)

$$1 - x = x - x^2$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0$$

$$x = 1 \Rightarrow y = 0$$

56. (A)

$$4y^2 + 6x = 8y + 7$$

$$(2y)^2 - 2 \times (2y) \times 2 + 4 = -6x + 11$$

$$(2y - 2)^2 = -6 \left(x - \frac{11}{6} \right)$$

$$(y - 1)^2 = \frac{-3}{2} \left(x - \frac{11}{6} \right)$$

Tangent at vertex $x = \frac{11}{6}$

57. (A)

$$4x^2 = 4x = -y + 2$$

$$(2x)^2 - 2 \cdot 2x \times 1 + 1 = -y + 3$$

$$(2x - 1)^2 = -(y - 3)$$

$$(x - \frac{1}{2})^2 = \frac{1}{4}(y - 3)$$

58. (D)

$$y^2 = x$$

$$y = x + c \quad (\tan 45^\circ = 1)$$

$$c = \frac{1/4}{1} = \frac{1}{4}$$

$$(x + 1/4)^2 = x$$

Point $(1/4, 1/2)$

59. (C)

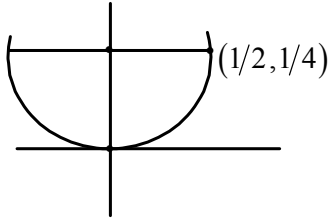
$$(2x + 2)^2 = 16x$$

$$(x + 1)^2 = 4x$$

$$(x - 1)^2 = 0$$

$$x = 1 \Rightarrow y = 4$$

60. (D)



$$x^2 = y$$

$$x^2 = 4 \times \left(\frac{1}{4}y\right)$$

Tangent at $(1/2, 1/4)$ is

$$x \times 1/2 = \frac{(y + 1/4)}{2}$$

$$4x - 4y = 1$$

61. (D)

Tangent of $y^2 = 4x$ is

$$y = mx + 1/m$$

Also for $x^2 = 32y$

$$x^2 = 32(mx + 1/m)$$

$$x^2 - 32mx - \frac{32}{m} = 0$$

For tangency $D = 0$

$$\Rightarrow (-32m)^2 + \frac{32}{m} \times 4 = 0$$

$$\Rightarrow m = -1/2$$

$$\therefore y = -\frac{1}{2}x - 2$$

$$\Rightarrow x + 2y + 4 = 0$$

62. ()

$$\Delta = \frac{1}{2} \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2} \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ a(t_2^2 - t_1^2) & 2a(t_2 - t_1) & 0 \\ a(t_3^2 - t_1^2) & 2a(t_3 - t_1) & 0 \end{vmatrix}$$

$$= \frac{1}{2} \times a^2 (t_2 - t_1)(t_3 - t_1)$$

$$\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ t_1 + t_2 & 2 & 0 \\ t_1 + t_3 & 2 & 0 \end{vmatrix}$$

$$\Delta = \frac{1}{2} a^2 (t_2 - t_1)(t_3 - t_1) \times 2(t_2 - t_3)$$

Required area is $1/2\Delta$

$$\therefore \text{Area} = \frac{a^2}{2} (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$$

63. ()

$$t, y = x + at_1^2 \text{ \& } t_2 y = x + at_2^2$$

Solving $x = at_1 t_2$

\therefore abscissa is G. M. between ponts.

EXERCISE 1(B)

1. (A)

$$at_1^2 = p$$

$$2ah = q$$

At conditions of θ be $\left(\frac{a}{h^2}, \frac{-2a}{h}\right)$

$$\text{Or } \left(\frac{a^2}{p}, \frac{-4a^2}{q}\right)$$

2. (A)

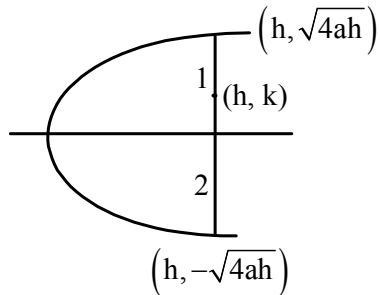
$$a + at^2 = 4$$

$$\text{Here } a = 2 \Rightarrow t^2 = 1$$

$$\Rightarrow t = \pm 1$$

Coordinate $(2, \pm 4)$

3. (C)



By section formula

$$\frac{2\sqrt{4ah} - \sqrt{4ah}}{3} = k$$

$$\frac{4ax}{J} = y^2$$

4. (B)

Check option (B)

$$x^2 - 2 = -2 \left(2 \cos^2 \frac{t}{2} - 1 \right)$$

$$x^2 - 2 = -4 \cos^2 \frac{t}{2} + 2$$

$$x^2 - 4 = -y$$

5. (C)

$$x_1 = at_1^2 \quad t_1 t_2 = -1$$

$$x_2 = at_2^2$$

$$x_1 x_2 = a^2$$

Square be G.M. of $x_1 x_2 = a^2$

6. (A)
 $y = mx + c$ is tangent to

$$y^2 = 4ax \Rightarrow c = \frac{a}{m}$$

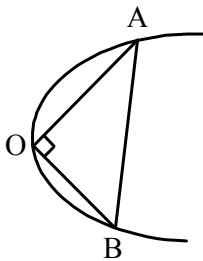
$$\& \text{ of } x^2 = 4by \Rightarrow c = -bm^2$$

$$\Rightarrow m^3 = -\frac{a}{b}$$

$$\text{Or } m = \frac{-a^{1/3}}{b^{1/3}}$$

$$\text{Equation of tangent is } a^{1/3}x + b^{1/3}y + (ab)^{2/3} = 0$$

7. (A)



Homogenize line with $y^2 = 4ax$

$$y^2 - 4ax \left(\frac{y + mx}{c} \right) = 0$$

Since $\angle AOB = 90^\circ$

$$\Rightarrow \text{coefficient } x^2 + \text{coefficient } y^2 = 0$$

$$c + 4ax = 0$$

8. (A)

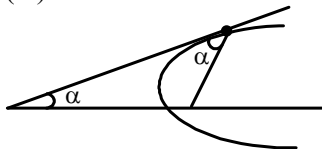
We write equation as

$$\left(\frac{2u^2 \cos^2 \alpha}{g} \right) y = \left(\frac{u^2 \sin 2\alpha}{g} \right) x - x^2$$

When we form a perfect square

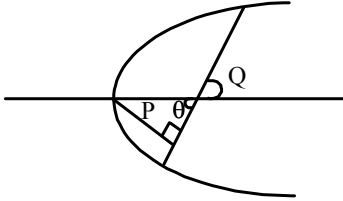
$$\text{We get } x = \frac{2u^2 \cos^2 \alpha}{g}$$

9. (A)



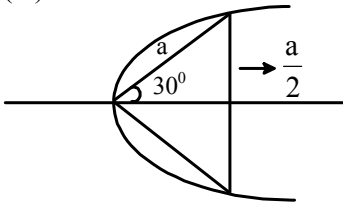
Properly these two angles are same.

10. (C)



$$P = a \sin \theta$$

11. (A)



The length of double ordinate will be a

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times a \cos 90^\circ \times a \\ &= \frac{\sqrt{3}}{4} a^2 \end{aligned}$$

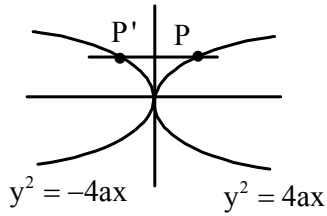
Here $P^2 \left(\frac{\sqrt{3}a}{2}, a/2 \right)$ lies on parabola

$$\begin{aligned} \frac{a^2}{4} &= \frac{4\sqrt{3}a}{2} \quad \therefore A = 48\sqrt{3} \\ a &= 8\sqrt{3} \end{aligned}$$

- | | | | | |
|---------|---------|---------|---------|---------|
| 12. (B) | 13. (D) | 14. (A) | 15. (C) | |
| 16. (B) | 17. (A) | 18. (C) | 19. (C) | 20. (D) |
| 21. (A) | 22. (A) | 23. (C) | 24. (D) | 25. (C) |
| 26. (D) | 27. (A) | 28. (C) | 29. (C) | 30. (C) |
| 31. (A) | 32. (D) | 33. (D) | 34. (C) | 35. (C) |
| 36. (C) | 37. (D) | 38. (B) | 39. (B) | 40. (C) |
| 41. (C) | 42. (B) | 43. (C) | 44. (A) | 45. (B) |
| 46. (A) | 47. (D) | 48. (B) | | |

(2-A) PARABOLA BOOKLET SOLUTION

1. (A)



$$P \equiv (at^2, 2at) \quad P' \equiv (-at^2, 2at)$$

Let unit point be $\theta(h, k)$

$$h = \frac{at^2 - bt^2}{2} \Rightarrow t^2 = \left(\frac{2h}{a-b} \right) \dots\dots\dots (I)$$

$$K = \frac{2at - 2bt}{2} \Rightarrow t = \left(\frac{K}{a-b} \right) \dots\dots\dots (II)$$

Eliminating t from (I) and (II)

$$\left(\frac{K}{a-b} \right)^2 = \frac{2h}{a-b}$$

$$\Rightarrow K^2 = 2h(a-b)$$

\therefore Locus is $y^2 = 2(a-b)x$. Which is a parabola

2. ()

$$y = m(x - a) \dots\dots\dots (I)$$

$$y^2 = 4ax \dots\dots\dots (II)$$

Solving (I) and (II)

$$[m(x - a)]^2 = 4ax$$

$$\Rightarrow m^2(x - a)^2 = 4ax$$

$$\Rightarrow m^2x^2 - 2am^2x + m^2a^2 = 4ax$$

$$\Rightarrow m^2x^2 - 2ax(m^2 + 2) + m^2a^2 = 0$$

For 2 real solutions $D > 0$ & $m^2 \neq 0$

$$\Rightarrow 4a^2(m^2 + 2)^2 - 4m^2(m^2a^2) > 0$$

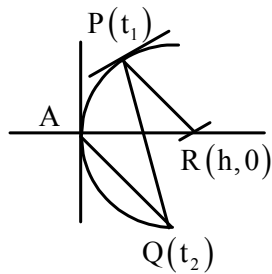
$$\Rightarrow 4a^2((m^2 + 2)^2 - m^4) > 0$$

$$\Rightarrow 4a^2((2m^2 + 2)(2)) > 0$$

$$\Rightarrow m \in \mathbb{R}$$

$$\therefore m \in \mathbb{R} \sim \{0\}$$

3. (C)



$\therefore PQ$ is a normal chord

$$\Rightarrow t_2 = -t_1 \frac{-2}{t_1} \dots\dots\dots (I)$$

Slope of $AQ = \frac{2}{t_2} =$ slope of PR .

Equation of PR .

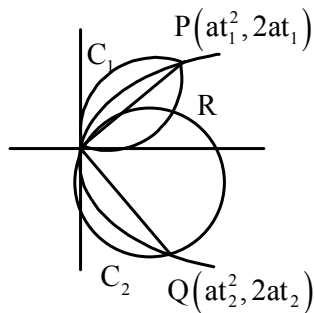
$$y - 2at_1 = \frac{2}{t_2}(x - at_1^2)$$

For R .

$$0 - 2at_1 = \frac{2}{t_2}(h - at_1^2)$$

$$\begin{aligned} \Rightarrow h &= at_1^2 - at_1 t_2 \\ &= at_1^2 - at_1 \left(-t_1 - \frac{2}{t_1}\right) \\ &= at_1^2 + at_1^2 + 2a = 2a(1 + t_1^2) \\ &= 2 \times \text{focal distance of } P. \end{aligned}$$

4. (D)



$$\tan \theta_1 = \frac{2}{t_1}$$

$$\tan \theta_2 = \frac{2}{t_2}$$

$$C_1 : x^2 + y^2 - at_1^2 x - 2at_1 y = 0$$

$$C_2 : x^2 + y^2 - at_2^2 x - 2at_2 y = 0$$

Equation of OR

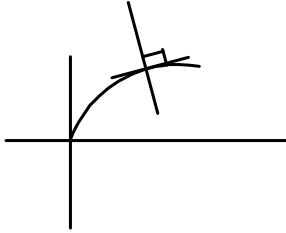
$$C_1 - C_2 = 0 \Rightarrow a(t_2^2 - t_1^2)x + 2a(t_2 - t_1)y = 0$$

$$\Rightarrow y = -\frac{1}{2}(t_1 + t_2)x$$

$$\therefore \tan \phi = -\frac{1}{2}(t_1 + t_2) = -\frac{1}{2}(\cot \theta_1 + \cot \theta_2)$$

$$\Rightarrow \cot \theta_1 + \cot \theta_2 = -2 \tan \phi$$

5. ()



$$y = \sqrt{x} \Rightarrow y^2 = x \quad \dots\dots\dots (I)$$

Equation of normal to (I) in parametric form.

$$y + tx = \frac{t}{2} + \frac{t^3}{4}$$

\therefore This passes through (3,6)

$$\Rightarrow 6 + 3t = \frac{t}{2} + \frac{t^3}{4} \Rightarrow t^3 - 10t - 24 = 0$$

$$\Rightarrow (t-4)(t^2 + 4t + 6) = 0$$

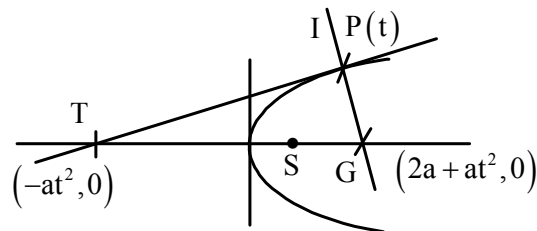
$$\Rightarrow t = 4$$

\therefore Normal is

$$y + 4x = 2 + 16$$

$$\Rightarrow 4x + y - 18 = 0$$

6. ()



Centre of circle passing through P, T, G is S(a,0)

\therefore Slope of tangent to circle at

$$P = \frac{-1}{\text{slope of PS.}}$$

$$= \frac{-1}{\left(\frac{2at}{at^2 - a}\right)} = \frac{1-t^2}{2t} = m_1$$

$$\text{Slope of tangent to parabola at P} = \frac{1}{t} = m_2$$

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| = \left| \frac{\frac{1}{t} - \frac{1}{2t} + \frac{t}{2}}{1 + \frac{1-t^2}{2t}} \right| = t$$

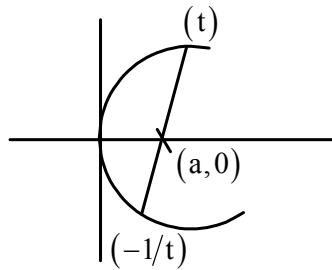
$$\therefore \theta = \tan^{-1}(t)$$

7. ()

Clearly the point of intersection of the tangents will always lie in the line. Hence locus will be the given line its effect.

8. ()

$$\text{Length of focal chord} = a \left(t + \frac{1}{t} \right)^2$$



$$\text{Also } \tan \alpha = \frac{2}{t - \frac{1}{t}}$$

$$\Rightarrow t \frac{-1}{t} = 2 \cot \alpha$$

$$\begin{aligned} \Rightarrow \left(t + \frac{1}{t} \right)^2 &= \left(t - \frac{1}{t} \right)^2 + 4 \\ &= 4 \cot^2 \alpha + 4 \\ &= 4 \operatorname{cosec}^2 \alpha \end{aligned}$$

$$\therefore \text{Length of focal chord} = 4 \operatorname{cosec}^2 \alpha$$

$$\because \alpha \in (0, \pi/4)$$

$$\therefore \text{Minimum length} = 4 \operatorname{cosec}^2 \frac{\pi}{4} = 8a$$

9. $y = k\sqrt{x} \Rightarrow y^2 = k^2 x$

Let $k^2 = 4a \Rightarrow y^2 = 4ax$

For ABCD

$$at_1^2 = 2at_1 \Rightarrow t_1 = 2$$

For EFGC

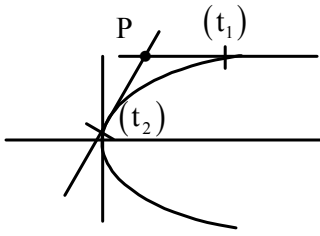
$$at_2^2 - ya = 2at_2$$

$$\Rightarrow at_2^2 - 2at_2 - 4a = 0$$

$$\Rightarrow t_2 = 1 + \sqrt{5}$$

$$\therefore \frac{FG}{BC} = \frac{2at_2}{2at_1} = \frac{2a(1+\sqrt{5})}{2a(2)} = \frac{\sqrt{5}+1}{2}$$

10. ()



Let $P \equiv (h, k)$ and pair of tangents drawn touches parabola at $A(t_1)$ and $B(t_2)$

$$\tan \theta_1 = \frac{1}{t_1} \quad \tan \theta_2 = \frac{1}{t_2}$$

$$\text{Also, } h = t_1 t_2 \quad k = (t_1 + t_2)$$

$$\text{Given, } \theta_1 + \theta_2 = \frac{\pi}{4}$$

$$\Rightarrow \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2} = 1$$

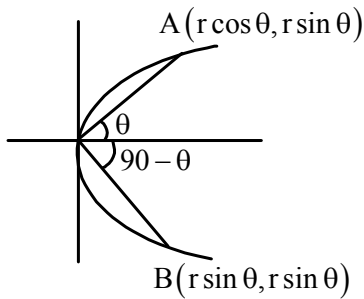
$$\Rightarrow \frac{\frac{1}{t_1} + \frac{1}{t_2}}{1 - \frac{1}{t_1 t_2}} = 1$$

$$\Rightarrow \frac{t_1 + t_2}{t_1 t_2 - 1} = 1$$

$$\Rightarrow \frac{k}{h-1} = 1 \Rightarrow h - k = 1$$

\therefore Locus is $x - y = 1$

11. ()



\therefore A and B lies on the parabola

For A

$$r^2 \sin^2 \theta = r \cos \theta$$

$$\Rightarrow r = \frac{\cos \theta}{\sin^2 \theta} = |VA|$$

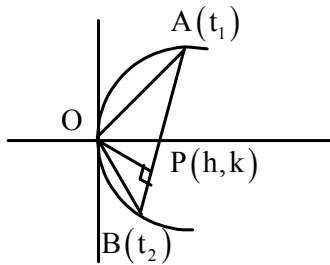
For B

$$r^2 \cos^2 \theta = r \sin \theta$$

$$\Rightarrow r = \frac{\sin \theta}{\cos^2 \theta} = |VB|$$

$$\therefore \frac{|VA|}{|VB|} = \frac{\cos^3 \theta}{\sin^3 \theta} = \cot^3 \theta$$

12. ()



\because AB subtends 90° at vertex

$$\Rightarrow t_1 t_2 = -4 \text{ ----- (I)}$$

Equation of AB

$$2x - y(t_1 + t_2) + 2at_1 t_2 = 0$$

$$\Rightarrow 2x - y(t_1 + t_2) - 8a = 0 \text{ (II)}$$

Equation of OP

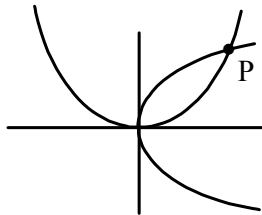
$$y = -\frac{(t_1 + t_2)}{2}x \text{ (III)}$$

To find locus of P eliminate t_1 and t_2 from (II) And (III)

Required locus is

$$x^2 + y^2 - 4ax = 0$$

13. ()



$$\text{Let } P \equiv (at^2, 2at)$$

\because P lies on $x^2 = 4by$ as well.

$$\Rightarrow a^2 t^4 = 4b \cdot 2at$$

$$\Rightarrow t = 2 \left(\frac{b}{a} \right)^{1/3}$$

Slope of tangent at P to on $y^2 = 4ax$

$$= \frac{2a}{y} = \frac{2a}{2at} = \frac{1}{t} \cdot (m_1)$$

Slope of tangent at P on $x^2 = 4by$

$$= \frac{x}{2b} = \frac{at^2}{2b} = \frac{a}{2b} \cdot 4 \cdot \frac{b^{2/3}}{a^{2/3}}$$

$$= 2 \left(\frac{a}{b} \right)^{1/3} = \frac{4}{t} (m_2)$$

\therefore Angle of intersection of parabola

=angle b/w their tangents

$$\begin{aligned} \therefore \tan \theta &= \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| = \left| \frac{\frac{4}{t} - \frac{1}{t}}{1 + \frac{4}{t^2}} \right| \\ &= \left| \frac{3}{t + \frac{4}{t}} \right| = \left| \frac{3/2}{\left(\frac{a}{b}\right)^{1/3} + \left(\frac{b}{a}\right)^{1/3}} \right| = \sqrt{3} \Rightarrow \theta = \pi/3 \end{aligned}$$

14. ()

Let point be P(h, k)

Equation of polar $xh + yk = r^2$

$$\Rightarrow y = \left(\frac{-h}{k}\right)x + \frac{r^2}{k}$$

For this to be tangent to $y^2 = 4ax$

$$\frac{r^2}{k} = \frac{a}{\left(\frac{-h}{k}\right)} \Rightarrow k^2 = -\frac{r^2}{a}h$$

$$\therefore \text{Locus is } y^2 = \frac{-r^2}{a}x$$

15. ()

$$x \cos \alpha + y \sin \alpha = P \Rightarrow y = (-\cot \alpha)x + P \operatorname{cosec} \alpha \quad \dots \dots \dots (I)$$

Tangent to $y^2 = 4a(x + a)$ will be

$$\begin{aligned} y &= m_1(x + a) + \frac{a}{m} \\ \Rightarrow y &= mx + am + \frac{a}{m} \quad \dots \dots \dots (II) \end{aligned}$$

Comparing (i) and (ii)

$m = -\cot \alpha$ and

$$\begin{aligned} P \operatorname{cosec} \alpha &= am + \frac{a}{m} \\ \Rightarrow P \operatorname{cosec} \alpha &= -a \cot \alpha - \frac{a}{\cot \alpha} \\ \Rightarrow P \operatorname{cosec} \alpha &= -\frac{a \operatorname{cosec}^2 \alpha}{\cot \alpha} \\ \Rightarrow P \cos \alpha + a &= 0 \end{aligned}$$

16. ()

Normal to $y^2 = 4c(x - d)$ can be

$$y = m(x - d) - 2cm - cm^3$$

$$\Rightarrow y = mx - md - 2cm - cm^3 \dots\dots (I)$$

Normal to $y^2 = 4ax$ can be

$$y = mx - 2am - am^3 \dots\dots\dots (II)$$

For common normal comparing (I) and (II)

$$-md - 2cm - cm^3 = -2am - am^3$$

$$\Rightarrow d + 2c + cm^2 = 2a + am^2$$

$$\Rightarrow (a - c)m^2 = (2c + d - 2a)$$

$$\because a > c > 0 \Rightarrow 2c + d - 2a > 0$$

$$\Rightarrow 2a < 2c + d$$

17. ()

Let normal be $y + tx = 2at + at^3$

$$\Rightarrow -t = \tan \phi \Rightarrow t = -\tan \phi$$

Normal cuts curve again at t_2

$$\Rightarrow t_2 = -t - \frac{2}{t} = \tan \phi + 2 \cot \phi$$

Slope of tangent at $t_2 = \frac{1}{t_2}$

Angle of intersection of normal at t_2

$$\begin{aligned} \tan \theta &= \left| \frac{\frac{1}{t_2} + t}{1 - \frac{t}{t_2}} \right| = \left| \frac{1 + tt_2}{t_2 - t_1} \right| = \left| \frac{-1 + t^2}{t_2 - t_1} \right| \\ &= \left| \frac{1 + \tan^2 \phi}{\tan \phi + 2 \cot \phi + \tan \phi} \right| = \left| \frac{\tan \phi}{2} \right| \end{aligned}$$

$$\therefore \theta = \tan^{-1} \left(\frac{1}{2} \tan \phi \right)$$

18. ()

Equation of focal chord

$$2x - y \left(t - \frac{1}{t} \right) - 2a = 0 \text{ and its length} = a \left(t + \frac{1}{t} \right)^2$$

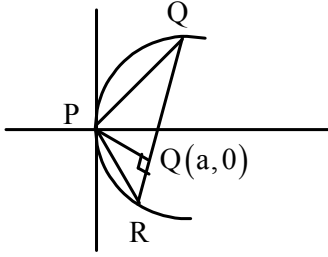
Distance of vertex focus this chord

$$P = \left| \frac{2a}{\sqrt{4 + (t - 1/t)^2}} \right| = \left| \frac{2a}{\sqrt{(t + 1/t)^2}} \right|$$

$$\Rightarrow \left(t + \frac{1}{t} \right)^2 = \frac{4a^2}{P^2}$$

$$\therefore \text{length of chord} = a \left(t + \frac{1}{t} \right)^2 = \frac{4a^3}{P^2}$$

19. ()



$$\text{Area of } \Delta = A = \frac{1}{2} \times a \left(t + \frac{1}{t} \right)^2 \times \frac{2a}{(t + 1/t)}$$

$$\left[\because QR = a \left(t + \frac{1}{t} \right)^2, PQ = \frac{2a}{\left(t + \frac{1}{t} \right)} \right]$$

$$\Rightarrow A = a^2 \left(t + \frac{1}{t} \right) \Rightarrow t + \frac{1}{t} = \frac{A}{a^2}$$

$$\text{Difference between ordinates} = 2at - \left(\frac{2a}{-t} \right) = 2a \left(t + \frac{1}{t} \right)$$

$$= 2a \frac{A}{a^2} = \frac{2A}{a}$$

20. ()

Same question as in Q. 3

21. ()

Slope of normal at $P(t_1)$ and $Q(t_2)$ is

$-t_1$ and $-t_2$ respectively

\because Normal are perpendicular to each other

$$\Rightarrow t_1 t_2 = -1$$

Equation of chord PQ

$$2x - y(t_1 + t_2) + 2at_1 t_2 = 0$$

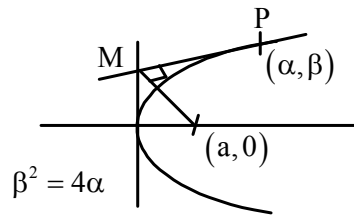
$$\Rightarrow 2(x - a) - (t_1 + t_2)(y) = 0$$

Which passes through intersection of lines

$$x - a = 0 \text{ and } y = 0$$

\therefore Fixed Pt is $(a, 0)$

22. ()



Equation of tangent at P

$$y\beta = 2a(x - \alpha)$$

Coordinates of M $\equiv \left(0, \frac{2a\alpha}{\beta}\right)$

$$\text{Area of } \Delta = \frac{1}{2} \begin{vmatrix} \alpha & \beta & 1 \\ 0 & \frac{2\alpha}{\beta} & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left(\frac{2\alpha^2}{\beta} + \left(\beta - \frac{2\alpha}{\beta} \right) \right) = \frac{1}{2} \left(\frac{\beta^3}{8} + \frac{\beta}{2} \right)$$

$$\text{Maximum area} = (\text{for } \beta = 2) = \frac{1}{2} \left(\frac{8}{8} + \frac{2}{2} \right) = 1$$

23. (A)

$$y^2 - 2y - 4x + 5 = 0$$

$$\Rightarrow (y-1)^2 = 4(x-1)$$

Equation of normal

$$y-1 = m(x-1) - 2m - m^3$$

Let point on axis of parabola be $(x, 1)$

$$\Rightarrow 0 = m(x-1) - 2m - m^3 \Rightarrow 0 = x - 3 - m^2$$

$$\Rightarrow m^2 = x - 3 \geq 0 \Rightarrow x \geq 3$$

24. ()

$$\text{Slope of line} = \sqrt{3} = \tan \theta \Rightarrow \theta = 60^\circ$$

Coordinate of a point at distance r from

$$(\sqrt{3}, 0). \text{ On the line } y - \sqrt{3}x + 3 = 0$$

$$\equiv (\sqrt{3} + r \cos 60^\circ, r \sin 60^\circ)$$

$$\equiv \left(\sqrt{3} + \frac{r}{2}, \frac{\sqrt{3}r}{2} \right)$$

Putting this on parabola

$$y^2 = x + 2$$

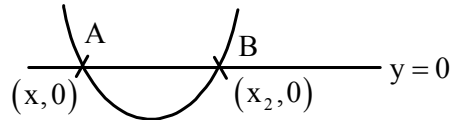
$$\Rightarrow \left(\frac{\sqrt{3}r}{2} \right)^2 = \left(\sqrt{3} + \frac{r}{2} \right) + 2$$

$$\Rightarrow \frac{3}{4}r^2 - \frac{r}{2} - (\sqrt{3} + 2) = 0$$

$$\text{PA.PB} = |\text{Product of roots}| = \frac{\sqrt{3} + 2}{3/4}$$

$$= \frac{4(2 + \sqrt{3})}{3}$$

25. ()



Equation of circle passing through A and B can be given by

$$(x - x_1)(x - x_2) + y^2 + \lambda y = 0$$

Where x_1 and x_2 are roots of $ax^2 + bx + c = 0$

\therefore Equation will be

$$x^2 + y^2 + \frac{b}{a}x + \lambda y + \frac{c}{a} = 0$$

$$\text{Length of tangent from origin} = \sqrt{\frac{c}{a}}$$

26. $\therefore (ap^2, 2ap), (aq^2, 2aq), (ar^2, 2ar)$ are

Co-normal point

$$\therefore p + q + r = 0$$

$$\Rightarrow x = 1 \text{ is a root of } px^2 + qx + r = 0$$

Also $x = 1$ satisfies

$$a(b - c)x^2 + b(c - a)x + c(a - b) = 0$$

\therefore common root is $x = 1$

27. ()

$$y = \frac{a^3}{3} \left(x^2 + \frac{3}{2a}x \right)$$

$$\Rightarrow \frac{3y}{a^3} = \left(x + \frac{3}{4a} \right)^2 - \frac{9}{16a^2}$$

$$\Rightarrow \left(\frac{3y}{a^3} + \frac{9}{16a^2} \right) = \left(x + \frac{3}{4a} \right)^2$$

$$\text{Vertex } h = \frac{-3}{4a} \quad k = \frac{-3a}{16}$$

$$hk = \frac{9}{64} \quad \therefore \text{locus is } xy = \frac{9}{64}$$

28. ()

Equation of tangent at (1,2)

$$2y = 2(x+1) \Rightarrow x - y + 1 = 0$$

Image of a variable point $(t^2, 2t)$ in the tangent

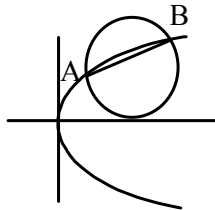
$$\frac{x - t^2}{1} = \frac{y - 2t}{-1} = \frac{-\cancel{2}(t^2 - 2t + 1)}{\cancel{2}}$$

$$\Rightarrow x = 2t - 1 \quad \& \quad y = t^2 + 1$$

$$\Rightarrow t = \left(\frac{x+1}{2}\right) \quad \& \quad t^2 = (y-1)$$

$$\left(\frac{x+1}{2}\right)^2 = (y-1) \Rightarrow (x+1)^2 = 4(y-1) \text{ is the required image}$$

29. ()



\because circle touches x - axis

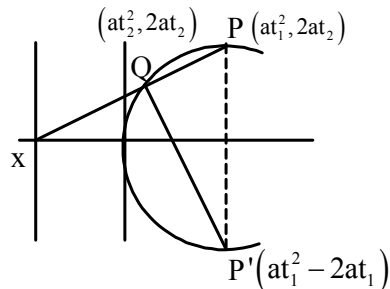
y coordinate of center = radius

$$\Rightarrow \frac{2at_1 + 2at_2}{2} = r$$

$$\Rightarrow t_1 + t_2 = \frac{r}{a}$$

$$\text{Slope of AB} = \frac{2}{t_1 + t_2} = \frac{2a}{r}$$

30. ()



Equation of PQ

$$2x - y(t_1 + t_2) + 2at_1t_2 = 0$$

\because It passes through $(-a, 0)$

$$\Rightarrow -2a + 0 + 2at_1t_2 = 0$$

$$\Rightarrow t_1t_2 = 1 \quad \dots\dots\dots (I)$$

$$\Rightarrow t_2 = \frac{1}{t_1}$$

$$\text{Equation of P'Q} \Rightarrow 2x - y(t_2 - t_1) - 2at_1t_2 = 0$$

$$\Rightarrow 2x - y(t_2 - t_1) - 2a = 0$$

$$\Rightarrow 2(x - a) - (t_2 - t_1)y = 0$$

Which passes through $(a, 0)$ i.e. focus

31. Let mid point of PQ be (h, k)

Equation of chord PQ

$$xh + yk = h^2 + k^2$$

$$\Rightarrow y = \left(\frac{-h}{k}\right)x + \frac{h^2 + k^2}{k}$$

For this to be tangent to parabola

$$c = a/m$$

$$\Rightarrow \frac{h^2 + k^2}{k} = \frac{a}{(-h/k)} \Rightarrow h(h^2 + k^2) + ak^2 = 0$$

\therefore locus is

$$x(x^2 + y^2) + ay^2 = 0$$

32. ()

Let mid point be $P(h, k)$

$$\text{Equation of chord} \Rightarrow xh - 2(y + k) = h^2 - 4k$$

$$\Rightarrow x = \left(\frac{2}{h}\right)y + \frac{h^2 - 2k}{h} \quad \dots\dots (I)$$

Equation of normal to $x^2 = 4y$

$$x = my - 2m - m^3 \quad \dots\dots\dots (II)$$

Comparing (I) and (II)

$$\frac{h^2 - 2k}{h} = -2\left(\frac{2}{h}\right) - \left(\frac{2}{h}\right)^3$$

$$\Rightarrow h^2 - 2k = -4 - \frac{8}{h^2} \Rightarrow 2k = h^2 + \frac{8}{h^2} + 4$$

$$\therefore \text{Locus is } 2y = x^2 + \frac{8}{x^2} + 4$$

33. ()

Equation of tangent to parabola $y^2 = 4ax$ is

$$yt = x + at^2 \quad \dots\dots\dots (I)$$

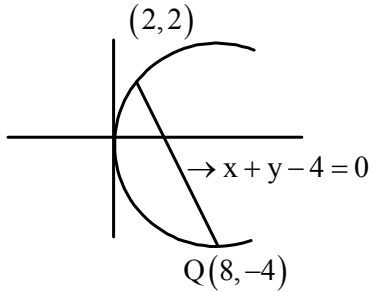
For (I) to be normal to circle $x^2 + y^2 - 2ax - 2by + c = 0$ it should pass through its centre (a, b)

$$\therefore bt = a + at^2 \Rightarrow at^2 - bt + a = 0$$

For 2 distinct tangent $D > 0$

$$\Rightarrow b^2 - 4a^2 > 0 \Rightarrow b^2 > 4a^2$$

34. ()



Point $(P^2, P-2)$ should lie inside the parabola as well as on the same side of the chord PQ as the origin

$$(P-2)^2 - 2P^2 < 0 \Rightarrow -P^2 - 4P + 4 < 0$$

$$\Rightarrow P^2 + 4P - 4 > 0$$

$$\Rightarrow P \in (-\infty, -2 - 2\sqrt{2}) \cup (-2 + 2\sqrt{2}, \infty) \quad \dots\dots\dots \text{(I)}$$

Also, $P^2 + P - 2 - 4 < 0 \Rightarrow P^2 + P - 6 < 0$

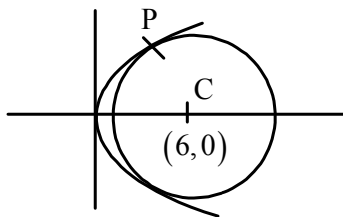
$$\Rightarrow (P+3)(P-2) < 0$$

$$\Rightarrow -3 < P < 2 \quad \dots\dots\dots \text{(II)}$$

$$\text{(I)} \cap \text{(II)}$$

$$\Rightarrow -2 + 2\sqrt{2} < P < 2$$

35. ()



For having a common tangent circle should touch the parabola

\therefore Its centre will lie on normal to the parabola at $P(t^2 - 2t)$

Substituting $(6, 0)$ in equation of normal at P, we get $t = 0, 2$ & -2

Least distance $CP = \sqrt{20}$

\therefore For no common tangent

$$CP > \text{radius}$$

$$\Rightarrow r < \sqrt{20}$$

36. ()

Equation of tangent to $y^2 = 4ax$ with slope m

$$y = mx + \frac{a}{m} \quad \dots\dots\dots \text{(I)}$$

Equation of normal to $x^2 = 4by$ of slope m.

$$y = mx + 2b + \frac{b}{m^2} \quad \dots\dots (II)$$

Comparing (I) and (II)

$$\frac{a}{m} = 2b + \frac{b}{m^2}$$

$$\Rightarrow 2bm^2 - am + b = 0$$

For 2 tangents $D > 0$

$$\Rightarrow a^2 - 8b^2 > 0 \Rightarrow b^2 < \frac{a^2}{8}$$

$$\Rightarrow |b| < \frac{|a|}{2\sqrt{2}}$$

37. ()

Let mid point of chord be $(at^2, 2at)$

Equation of chord

$$x.at^2 + y.2at = a^2t^4 + 4a^2t^2$$

\therefore It passes through (a, a)

$$\Rightarrow a^2t^2 + 2a^2t = a^2t^4 + 4a^2t^2$$

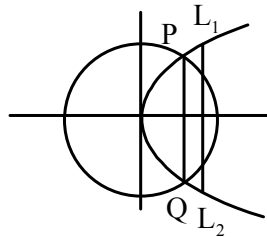
$$\Rightarrow t(t^3 + 3t^2 - 2) = 0$$

$t = 0$ and one real root from $t^3 + 3t^2 - 2 = 0$

\therefore 2 values of t .

\therefore 2 chords are possible

38. ()



$$P \equiv (at^2, 2at)$$

Equation of circle

$$x^2 + y^2 = \frac{9a^2}{4}$$

\therefore P lies on circle also

$$\Rightarrow (at^2)^2 + (2at)^2 = \frac{9}{4}a^2$$

$$\Rightarrow 4t^4 + 16t^2 - 9 = 0 \Rightarrow (2t^2 + a)(2t^2 - 1)$$

$$\Rightarrow t = \pm \frac{1}{\sqrt{2}}$$

$$PQ = 4at = 2\sqrt{2}a$$

$$L_1L_2 = 4a$$

$$\text{Distance between PQ and } L_1L_2 = \frac{a}{2}$$

$$\therefore \text{ area} = \frac{1}{2} \times \frac{a}{2} (4a + 2\sqrt{2}a) = a^2 \left(\frac{2 + \sqrt{2}}{2} \right)$$

39. ()

Equation of normal

$$y + tx = 2at + at^3$$

A $\equiv (\alpha, \beta)$ lies on it

$$\Rightarrow \beta + t\alpha = 2at + at^3$$

$$\Rightarrow at^3 + (2a - \alpha)t - \beta = 0 \quad \dots\dots\dots (I)$$

$$at^3 + (2a - \alpha)t = \beta$$

Squaring

$$a^2t^6 + (2a - \alpha)^2t^2 + 2a(2a - \alpha)t^4 = \beta^2$$

$$\text{Focal distance} = sp = r = a(1 + t^2)$$

$$\Rightarrow t^2 = \left(\frac{r - a}{a} \right)$$

\therefore Equation becomes

$$a^2 \left(\frac{r - a}{a} \right)^3 + (2a - \alpha)^2 \left(\frac{r - a}{a} \right) + 2a(2a - \alpha) \left(\frac{r - a}{a} \right)^2 - \beta^2 = 0$$

SP. SQ. SR = (Product of roots)

$$\begin{aligned} &= - \left(\frac{-a^2 - (2a - \alpha)^2 + 2a(2a - \alpha) - \beta^2}{\frac{1}{a}} \right) \\ &= -a(-a^2 - 4a\alpha - \alpha^2 + 4a^2 - 2a\alpha - \beta^2) \\ &= -a(-a^2 + 2a\alpha - \alpha^2 - \beta^2) \\ &= a(a^2 - 2a\alpha + \alpha^2 + \beta^2) \\ &= a((\alpha - a)^2 + \beta^2) \\ &= a(SA)^2 \\ \therefore I &= a \end{aligned}$$

40. P \equiv (h, k)

Equation of chord of contact

$$yk = 2a(x + h)$$

$$\Rightarrow y = \left(\frac{2a}{k}\right)x + \left(\frac{2ah}{k}\right) \dots\dots\dots (I)$$

Equation of tangent to $x^2 = 4by$

$$y = mx - bm^2 \dots\dots\dots (II)$$

From (I) and (II)

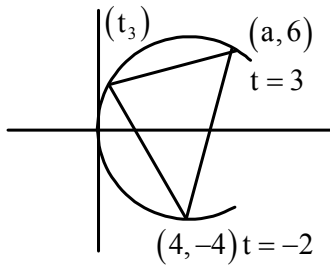
$$\frac{2ah}{k} = -\left(\frac{2a}{k}\right)^2$$

$$\Rightarrow 2akh = -4a^2b$$

$$\Rightarrow \text{locus is } xy = -2ab$$

i.e. a hyperbola

41. ()



Substituting P and Q in the parabola we get $a = 1, b = 0$

$$\text{Area of } \Delta = a^2 |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

$$= a^2 |(t_3 + 2)(t_3 - 3)(5)|$$

$$= 5a^2 |t^2 - t - 6|$$

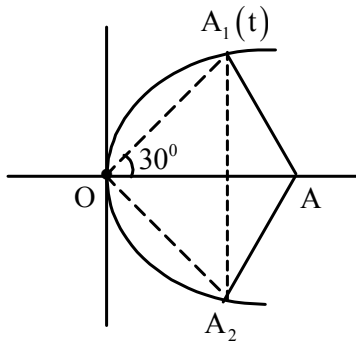
$$= 5a^2 \left| (t - 1/2)^2 - \frac{25}{4} \right|$$

$$\text{Area} = 5a^2 \left(\frac{25}{4} - (t - 1/2)^2 \right)$$

Area will maximum when $t = 1/2$

$$\therefore \text{Pt is } \left(\frac{1}{4}, \frac{1}{2}\right)$$

42. ()



For OAA_1 to be equilateral

$$\frac{2}{t} = \tan 30^\circ = \frac{1}{\sqrt{3}} \Rightarrow t = 2\sqrt{3}$$

Also, equation of normal $y + tx = 2at + at^{-3}$

\therefore A lies on it

$$\Rightarrow ht = 4t + 2t^3$$

$$\Rightarrow h = 4 + 2t^2$$

$$= 4 + 2 \times 12 = 28$$

- 21) Hint: Common tangent concept.
- 22) Hint: Apply condition of tangency and check options.

23) $y^2 = 2px$ — (1)
 focus $\equiv (\frac{p}{2}, 0)$, directrix $x = -\frac{p}{2}$
 $\Rightarrow 2x + p = 0$

so circle is
 $(x - \frac{p}{2})^2 + y^2 = r^2$ — (2)
 As circle touches the directrix $2x + p = 0$, so

Applying condⁿ of tangency

$$\left| \frac{2 \cdot \frac{p}{2} + p}{2} \right| = r$$

$$|p| = r$$

so eqⁿ (2) becomes

$$(x - \frac{p}{2})^2 + y^2 = p^2$$
 — (3)

solving eqⁿ (1) and (3) simultaneously
options

- 24) Hint: direct property.
- 25) Hint: use concept of perpendicular chords.
- 26) Hint: make the diagram using given information.

(27) Hint: use concept of Normals and property.

28

$$\text{Let } (x_1, y_1) \equiv (at^2, 2at)$$

Tangent at this point is $ty = x + at^2$

Any point on this tangent is $(h, \frac{h+at^2}{t})$

Chord of contact of this point w.r. to. the circle

$$x^2 + y^2 = a^2 \text{ is}$$

$$hx + \left(\frac{h+at^2}{t}\right)y = a^2$$

$$\text{or } (aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$$

which is a family of st. lines passing through the point of intersection of

$$aty - a^2 = 0$$

and

$$x + \frac{y}{t} = 0$$

} fixed point is $\left(-\frac{a}{t^2}, \frac{a}{t}\right)$

$$\therefore x_2 = -\frac{a}{t^2}, y_2 = \frac{a}{t}$$

$$\text{clearly } x_1 x_2 = -a^2, y_1 y_2 = 2a^2$$

$$\text{Also } \frac{y_1}{x_2} = -t^4$$

$$\frac{y_1}{y_2} = 2t^2$$

$$\Rightarrow 4 \frac{y_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$$

(29)

$$t_2 = -t_1 - \frac{2}{t_1}$$

5d

$$\text{Also } \frac{2at_1}{at_1^2} \times \frac{2at_2}{at_2^2} = -1$$

$$\Rightarrow t_1 \cdot t_2 = -4$$

$$\therefore -\frac{4}{t_1} = -t_1 - \frac{2}{t_1}$$

$$\Rightarrow t_1^2 + 2 = 4 \Rightarrow t_1 = \pm\sqrt{2}$$

So point can be $(2a, \pm 2a\sqrt{2})$

(30) As a Circle can intersect a parabola in 4 points
AB \rightarrow so, quadrilateral may be cyclic.

The diagonals of the quadrilateral may be equal if the quadrilateral may be an isosceles trapezium.



A rectangle can not be inscribed in a parabola, so (C) is wrong.

(31)

(32)

(33)

(34)

~~At~~ Concept Question

(31) Any point on the line $x+y=1$ is $(h, 1-h)$.
 eqⁿ of chord whose mid point is known w.r. to y
 $= 4ax$

is $T=S_1$

$$(1-h)y - 2a(x+h) = (1-h)^2 - 4ah \quad \text{--- (1)}$$

eqⁿ (1) is passing through $(a, 2a)$, so.

$$(1-h)2a - 2a(a+h) = (1-h)^2 - 4ah$$

$$\Rightarrow 2a - 2ah - 2a^2 - 2ah = h^2 - 2h + 1 - 4ah$$

$$\Rightarrow h^2 - 2h + 1 + 2a^2 - 2a = 0$$

$$D > 0 \Rightarrow B^2 - 4AC > 0$$

$$4 - 4(1)(1 + 2a^2 - 2a) > 0$$

$$4 - 4 - 2a^2 + 2a > 0$$

$$-2a(a-1) > 0$$

$$\Rightarrow a(a-1) < 0$$

$$\Rightarrow 0 < a < 1$$

$$\Rightarrow \underline{0 < 4a < 4}$$

so L.R. can be (A, B, D) .

(32) direct (solve both eqns and use condⁿ of common tangency)

(33) $y^2 = 4x$

Normal to the given parabola with slope m is

$$y = mx - 2m - m^3 \quad \text{--- (1)}$$

eqⁿ (1) is passing through $(9, 6)$, so

$$6 = 9m - 2m - m^3$$

$$\Rightarrow \cancel{m^3} - \cancel{6m} - m^3 - 7m + 6 = 0$$

$$\Rightarrow (m-1)(m^2+m-6) = 0$$

$$\Rightarrow (m-1)(m+3)(m-2) = 0$$

$$\Rightarrow m = 1, -3, 2$$

options A, B, D.

(34) $A \equiv (at_1^2, 2at_1)$

$$B \equiv (at_2^2, 2at_2)$$

$$OA \perp OB \Rightarrow t_1 t_2 = -4$$

find OA and OB using distance formula.

Substitute 2
 $OA = r_1^2$

$$OB = r_2^2$$

then simplify the expansion.

Passages

Any parabola whose axes is parallel to x-axis will be of the form

$$(y-a)^2 = 4b(x-c) \quad \text{--- (1)}$$

Now letting eq (1) can be rewritten as

$$y-a = -\frac{l}{m}(x-c) + \frac{1-lc-am}{m} \quad \text{--- (2)}$$

eqⁿ (2) will touch (1) if

$$\frac{1-am-lc}{m} = -\frac{b}{l/m}$$

$$-\frac{l}{m} = \frac{bm}{1-am-lc}$$

$$\Rightarrow cl^2 - bm^2 + al - l = 0 \quad \text{--- (3)}$$

$$\text{But given that } 5l^2 + 6m^2 - 4lm + 3l = 0 \quad \text{--- (4)}$$

Combining (3) and (4), we get,

$$\frac{c}{5} = -\frac{b}{6} = \frac{a}{-4} = -\frac{1}{3}$$

$$\Rightarrow c = -\frac{5}{3}, b = 2 \text{ and } a = \frac{4}{3}$$

So parabola is

$$\left(y - \frac{4}{3}\right)^2 = 8\left(x + \frac{5}{3}\right) \text{ whose focus is } \left(\frac{1}{3}, \frac{4}{3}\right) \text{ and directrix is } 2x + 11 = 0$$

Passage-6 (b, c, d)

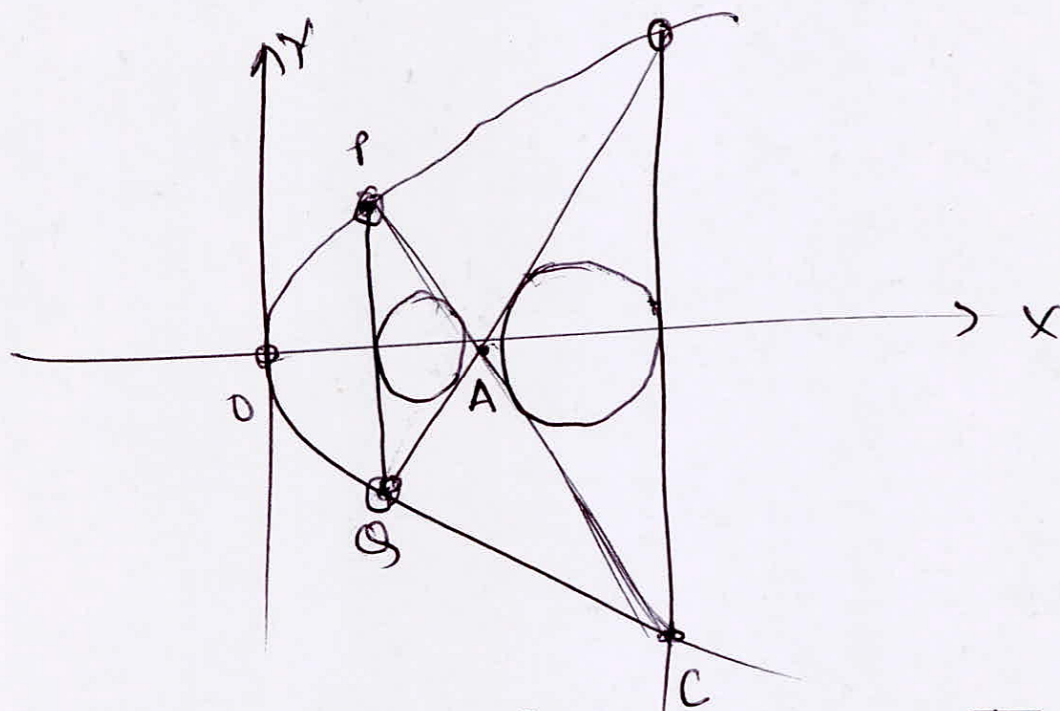
(2)

Solving given parabolas

$$-8(x-a) = 4x \Rightarrow x = \frac{2a}{3}$$

\Rightarrow points of intersection are $\left(\frac{2a}{3}, \pm\sqrt{\frac{8a}{3}}\right)$

Now OABC is concyclic.



$\triangle APB$ is an isosceles right angled. Therefore, slope of PA is -1 and its equation is

$$y-2 = -(x-1) \text{ or } x+y=3$$

Similarly eqⁿ of line BC is $x-y-3=0$

Solving $x+y=3$ with the parabola $y^2=4ax$, we

have $(3-x)^2 = 4x$ or $x^2 - 10x + 9 = 0 \Rightarrow x=1, 9$

Therefore co-ord. of B and C are $(1, 2)$ and $(9, 6)$

res p.

$$\therefore \text{Area of Trapezium } PBCQ = \frac{1}{2} (12 + 4) \times 8$$

$$= 64 \times 9 \text{ Units}$$

Let the Circumcentre of trapezium $PBCQ$ is $T(h, 0)$. $PT = TB$

$$\sqrt{(h-1)^2 + 4} = \sqrt{(h-9)^2 + 36}$$

$$\Rightarrow h = 7$$

Hence radius is $\sqrt{40} = 2\sqrt{10}$

Let the inradius of $\triangle PAB$ be r_1 , then $r_1 = \frac{\Delta_1}{s_1}$

$$r_1 = \frac{\frac{1}{2} \times 4 \times 2}{4 + 2\sqrt{4+4}} = \sqrt{2} - 1$$

Let inradius of $\triangle ABC$ be r_2 , then

$$\begin{aligned} r_2 &= \frac{\Delta_2}{s_2} = \frac{\frac{1}{2} \times 12 \times 6}{12 + 2\sqrt{36+36}} \\ &= \frac{3}{1+\sqrt{2}} = 3(\sqrt{2}-1) \end{aligned}$$

So $\frac{r_2}{r_1} = 3$

Passage-7

$$|z-5-3i| = \frac{z+\bar{z}-6}{2}$$

Um $z = x+iy$

on simplification it gives

$$(y-3)^2 = 4(x-4)$$

with L.R. = 4, Vertex (4,3)

$$\text{focus} = (5,3)$$

Based on the above information options are

(53) C,

(54) A

(55) B

Passage-8

A line through P(6,5) having slope m is

$$y-5 = m(x-6)$$

$$\Rightarrow y = mx + 5 - 6m \quad \text{--- (1)}$$

eqⁿ (1) is a tangent to $y^2 = 4x$ so,

$$5 - 6m = \frac{1}{m} \Rightarrow 6m^2 - 5m + 1 = 0$$

$$m = 1/2, 1/3$$

so tangents PQ and PR are,

PQ: $2y = x + 4$

PR: $3y = x + 9$

so solving ~~PR with~~ tangent PR with $y^2 = 4x$
point R is $(9, 6)$.

Now eqⁿ of a family of circles touching a given
~~line~~ line PR at a given point R is.

$$(x-9)^2 + (y-6)^2 + \lambda(3y-x-9) = 0 \quad \text{--- (2)}$$

eqⁿ (2) is passing through focus $(1, 0)$ so,

$$64 + 36 + \lambda(-10) = 0$$

$$\lambda = 10$$

eqⁿ (2) becomes

$$x^2 + y^2 - 28x + 18y + 27 = 0$$

whose radius is $5\sqrt{10}$.

Based on above information

Q. 56 — (A)

Q. 57 — (B)

Q. 58 — (C)

Solutions
PARABOLA
Ex. 3

Q.1

Let P be (h, k). Also let tangents from P be $t_1y = x + at_1^2$ & $t_2y = x + at_2^2$, where points of contact of these tangents being $Q(at_1^2, 2at_1)$ & $R(at_2^2, 2at_2)$.

Now point of intersection of tangents will be $h = at_1t_2, k = a(t_1 + t_2)$. Area of triangle PQR will now be given by

$$\frac{1}{2} \begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_1t_2 & a(t_1 + t_2) \end{vmatrix} = 4a^2 \text{ which implies } (t_1 - t_2)^2 = 4.$$

But $h = at_1t_2, k = a(t_1 + t_2) \Rightarrow a^2(t_1 - t_2)^2 = k^2 - 4ah$, hence $k^2 - 4ah = 16a^2$.

Required locus is $y^2 = 4a(x - 4a)$ which is a parabola.

Q.2

Let P & Q be $(at_1^2, 2at_1)$ & $(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.

Now $OQ^2 = a(t_2^4 + 4t_2^2)$ or $OQ^2 = a^2 \left[\left(t_1 + \frac{2}{t_1} \right)^4 + 4 \left(t_1 + \frac{2}{t_1} \right)^2 \right]$

$$\Rightarrow OQ^2 = a^2 \left[\left(t_1 + \frac{2}{t_1} \right)^2 + 2 \right]^2 - 4a^2. \text{ But by A.M. } \geq \text{ G.M., } \left| t_1 + \frac{2}{t_1} \right| \geq 2\sqrt{2}.$$

$$\Rightarrow \left[\left(t_1 + \frac{2}{t_1} \right)^2 + 2 \right]^2 \geq 100. \text{ Hence } |OQ| \geq 4a\sqrt{6}.$$

Q.3

If normal at $P(t_1)$ & (t_2) meet on the parabola, then $t_1t_2 = 2$.

Also P, Q, R & N (point of intersection of normals) will form a cyclic quadrilateral and circle passing through P, Q & R will have RN as diameter as $\angle PRN = \frac{\pi}{2}$.

Now coordinates of R will be $(at_1t_2, a(t_1 + t_2))$ or $(2a, a(t_1 + t_2))$. Similarly coordinates of N will be $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$ or $(a(t_1^2 + t_2^2 + 4), -2a(t_1 + t_2))$

Now let the circum center be (h, k), then

$$h = \frac{a(t_1^2 + t_2^2 + 6)}{2} \text{ \& } k = -\frac{a(t_1 + t_2)}{2} \Rightarrow \frac{2h}{a} - 6 = t_1^2 + t_2^2 \text{ \& } \frac{4k^2}{a^2} = t_1^2 + t_2^2 + 2t_1t_2$$

Or eliminating t gives & replacing (h, k) with (x, y) gives required locus as $2y^2 = a(x - a)$.

Q.4

Substituting $y = ax^2 - b$ in $x^2 + y^2 = 1$ gives $x^2 + (ax^2 - b)^2 = 1$ or $a^2x^4 + (1 - 2ab)x^2 + b^2 - 1 = 0$.

Now for four distinct points of intersection the above equation must have four distinct real roots. As the given equation is a biquadratic so considering $x^2 = t$ gives a quadratic in t both of whose roots must be real & positive.

Hence $a^2, 2ab - 1, b^2 - 1$ must be of same sign and $(1 - 2ab)^2 > 4a^2(b^2 - 1)$.

$$\supset 2ab > 1, b > 1, 4a^2 - 4ab + 1 > 0.$$

Clearly if $a > b > 1$, then all the above conditions get satisfied.

(remember here that $a > b > 1$ is a sufficient condition and may not be necessary)

Q.5

Let P, Q, P' & Q' be $(at_1^2, 2at_1), (at_2^2, 2at_2), (at_3^2, 2at_3)$ & $(at_4^2, 2at_4)$.

Now PQ is a focal chord & PP', QQ' are normal chords hence $t_2 = -\frac{1}{t_1}, t_3 = -t_1 - \frac{2}{t_1}$ & $t_4 = \frac{1}{t_1} + 2t_1$.

Slope of PQ = $\frac{2t_1 - 2t_2}{t_1^2 - t_2^2}$ or $\frac{2t_1}{t_1^2 - 1}$. Similarly

Slope of P'Q' = $\frac{2t_4 - 2t_3}{t_4^2 - t_3^2}$ or $\frac{2t_1}{t_1^2 - 1}$, hence PQ is parallel to P'Q'.

Also PQ = $a\left(t_1 + \frac{1}{t_1}\right)^2$ & P'Q' = $a\sqrt{(t_3^2 - t_4^2)^2 + 4(t_3 - t_4)^2}$ or $a|t_3 - t_4|\sqrt{(t_3 + t_4)^2 + 4}$

$\Rightarrow P'Q' = 3a\left(t_1 + \frac{1}{t_1}\right)^2$, hence P'Q' = 3 PQ.

Q.6

Let the fixed point on axis be P(h, 0), then any line passing through this point will be $y = m(x - h)$.

Substituting $(at^2, 2at)$ this gives $amt^2 - 2at - hm = 0$.

$\supset t_1 + t_2 = \frac{2}{m}$ & $t_1 t_2 = -\frac{h}{a}$, where t_1 & t_2 are parameters of those points where this line meets the

parabola. Also $t_1^2 + t_2^2 = \frac{4}{m^2} + \frac{2h}{a}$.

Now circle having this chord as diameter will be

$$x^2 + y^2 - a(t_1^2 + t_2^2)x - 2a(t_1 + t_2)y + a^2 t_1^2 t_2^2 + 4a^2 t_1 t_2 = 0.$$

$$\text{Or } x^2 + y^2 - a\left(\frac{4}{m^2} + \frac{2h}{a}\right)x - \frac{4a}{m}y + h^2 - 4ah = 0.$$

Now if we consider two such circles with $m = m_1$ & $m = m_2$, then radical axis of these circles will be

$$\left(\frac{1}{m_1^2} - \frac{1}{m_2^2}\right)x + \left(\frac{1}{m_1} - \frac{1}{m_2}\right)y = 0 \quad \text{or} \quad (m_1 + m_2)x + m_1 m_2 y = 0.$$

Clearly it passes through the origin.

Q.7

Comparing $P(16, 16)$ with $(4t^2, 8t)$ gives $t = 2$.

Now tangent at P will be $2y = x + 16$ & normal at P will be $2x + y = 48$.

Points where these lines meet the x-axis will be $A(-16, 0)$ & $B(24, 0)$.

As angle APB is a right angle hence the circle passing through P, A & B will have AB as diameter.

Hence $C_1 : (x + 16)(x - 24) + y^2 = 0$.

Equation of common chord of C_1 & C_2 will be $6x + y + 197 = 0$.

Q.8

Let $l = at^2$ & $m = 2at$. Now vertices of the triangle are $A(0, 2)$, $B\left(0, \frac{1}{2at}\right)$ & $C\left(\frac{1-4at}{at^2}, 2\right)$.

As the triangle is right angled hence by the concept of Euler's line its circum center (x, y) will be

$$\left(0 + 0 + \frac{1-4at}{at^2}, 2 + \frac{1}{2at} + 2\right).$$

Now $t = \frac{1}{2a(y-4)} \Rightarrow x = 4a(y-6)(y-4)$, which is equation of a parabola.

Q.9

Any tangent to $y^2 = 4ax$ will be $y = mx + \frac{a}{m}$ and any normal to $x^2 = 4by$ will be $y = mx + 2b + \frac{b}{m^2}$

Comparing the two equations gives $\frac{a}{m} = 2b + \frac{b}{m^2}$ or $2bm^2 - am + b = 0$.

For this equation to have real & distinct roots $a^2 > 8b^2$.

Q.10

Let B & C be $(at_1^2, 2at_1)$ & $(at_2^2, 2at_2)$ such that A is $(at_1t_2, a(t_1 + t_2))$. Also let another tangent be

drawn at $D(at_3^2, 2at_3)$ such that P & Q are $(at_1t_3, a(t_1 + t_3))$ & $(at_2t_3, a(t_2 + t_3))$.

Now $AP = a|t_2 - t_3|\sqrt{t_1^2 + 1}$, $AQ = a|t_3 - t_1|\sqrt{t_2^2 + 1}$.

Also $AB = a|t_2 - t_1|\sqrt{t_1^2 + 1}$ & $AC = a|t_2 - t_1|\sqrt{t_2^2 + 1}$.

$$\Rightarrow \frac{AP}{AB} + \frac{AQ}{AC} = \frac{|t_2 - t_3| + |t_3 - t_1|}{|t_2 - t_1|}.$$

Now considering t_1, t_3 & t_2 in cyclic order we get $\frac{AP}{AB} + \frac{AQ}{AC} = 1$.

Q.11

Let the point K be $(h, 0)$ and slope of chord through K be $\tan \theta$, then any point on this line at a distance r from K will be $(h + r \cos \theta, r \sin \theta)$.

For $r = PK$ & $r = QK$, this point will satisfy the equation of parabola, hence by substituting these coordinates in the equation of the parabola we get $(\sin^2 q)r^2 - (4a \cos q)r - 4ah = 0$.

Roots of this equation are PK & $-QK$, hence $PK - QK = \frac{4a \cos q}{\sin^2 q}$, & $PK \cdot QK = \frac{4ah}{\sin^2 q}$.

$$\text{Now } \frac{1}{PK^2} + \frac{1}{QK^2} = \frac{(PK - QK)^2 + 2PK \cdot QK}{(PK \cdot QK)^2} \supset \frac{1}{PK^2} + \frac{1}{QK^2} = \frac{16a^2 \cos^2 \theta + 8ah \sin^2 \theta}{64a^2 h^2}$$

$$\text{Clearly if } h = 2a, \text{ then } \frac{1}{PK^2} + \frac{1}{QK^2} = \frac{1}{4h^2}.$$

Q.12

$$\text{Any tangent to } y^2 = 4a(x + a) : y = mx + am + \frac{a}{m}$$

$$\& \text{ an orthogonal tangent to } y^2 = 4b(x + b) : y = -\frac{1}{m}x - \frac{b}{m} - bm.$$

Arranging both the equations as quadratic equations in m gives

$$bm^2 + my + x + b = 0 \& (x + a)m^2 - ym + a = 0.$$

$$\text{Comparing the two equations gives } \frac{b}{x + a} = \frac{y}{-y} = \frac{x + b}{a} \supset x + a + b = 0.$$

Now combining $y^2 = 4a(x + a)$ & $y^2 = 4b(x + b)$ in order get a linear equation we get common chord as $x + a + b = 0$.

Q.13

Let the fixed parabola be $y^2 = 4ax$ & the variable parabola be $(y - k)^2 = -4a(x - h)$ having vertex at $P(h, k)$.

Now as the two parabolas touch hence tangent to the two parabolas at $(at^2, 2at)$ must be the same.

$$\text{Tangent to } y^2 = 4ax \text{ will be } y = mx + \frac{a}{m} \text{ touching it at } T\left(\frac{a}{m^2}, \frac{2a}{m}\right) \&$$

$$\text{that to } (y - k)^2 = -4a(x - h) \text{ will be } y - k = m(x - h) - \frac{a}{m} \text{ or } y = mx + k - mh - \frac{a}{m}.$$

Comparing the two equations gives $k - mh = \frac{2a}{m}$ & substituting coordinates of T in equation of

$$\text{variable parabola gives } \left(\frac{2a}{m} - k\right)^2 = -4a\left(\frac{a}{m^2} - h\right).$$

$$\text{Or } hm^2 - km + 2a = 0 \& (k^2 - 4ah)m^2 - 4akm + 8a^2 = 0.$$

Comparing the two equations in order to eliminate m gives

$$\frac{k^2 - 4ah}{h} = 4a \text{ or } k^2 = 8ah, \text{ hence required locus is } y^2 = 8ax.$$

Q.14

Adding the two equations gives $x^2 + 6x - 4y + 13 = 0$ or $(x + 3)^2 = 4(y - 1)$, which means each of the points A, B, C & D lie on a parabola with vertex at $(-3, 1)$ and focus at $P(-3, 2)$.

Hence PA, PB, PC, PD will be focal distances of these points.

Now let any point on this parabola be $(2t - 3, t^2 + 1)$. Substitute these coordinates in the equation

$$x^2 - y^2 + 6x + 16y - 46 = 0 \text{ to get } t^4 - 18t^2 + 24t + 40 = 0.$$

Now let the roots of this be t_1, t_2, t_3, t_4 , then

$$t_1 + t_2 + t_3 + t_4 = 0, t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = -18$$

Also Focal distance of a point with parameter t will be $1 + t^2$, hence

$$PA + PB + PC + PD = 4 + t_1^2 + t_2^2 + t_3^2 + t_4^2.$$

Now from above relations

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = (t_1 + t_2 + t_3 + t_4)^2 - 2(t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4)$$

Therefore $PA + PB + PC + PD = 40$.

Q.15

Normal to $y^2 = 4ax$ at any point $P(t)$ will be $tx + y = 2at + at^3$.

This will meet the x-axis at $Q(2a + at^2, 0)$.

The line perpendicular to normal and passing through Q will be $x - ty = 2a + at^2$.

Now this equation may be rearranged as $y = m(x - 2a) - \frac{a}{m}$, where $m = \frac{1}{t}$.

Clearly its in form of tangent line of slope m to the parabola $y^2 = -4a(x - 2a)$.

Q.16

Let mid point of any such chord be $M(at^2, 2at)$.

Now using $T = S_1$, equation of chord of $x^2 + y^2 = 16a^2$ having mid point at M may be represented as $at^2x + 2aty = a^2t^4 + 4a^2t^2$.

As this chord is drawn through $(h, 0)$ hence substituting these coordinates in equation of chord we get $at^2h + 2aty = a^2t^4 + 4a^2t^2$.

Now the above equation gives three values of t , namely 0 & $\pm\sqrt{\frac{h-4a}{a^2}}$ out of which the later two values will be real & other than 0 only if $h > 4a$.

Also for M to be mid point of chord it must lie inside the circle hence

$$a^2t^4 + 4a^2t^2 - 16a^2 < 0 \text{ or } \left(\frac{4a-h}{a^2}\right)^2 + 4\left(\frac{4a-h}{a^2}\right) - 16 < 0, \text{ hence } h < (\sqrt{5} + 1)2a.$$

Q.17

Let A, B & P be $(at_1^2, 2at_1), (at_2^2, 2at_2)$ & $(at_1t_2, a(t_1 + t_2))$ on $y^2 = 4ax$.

Tangent PB will be $x - t_2y + at_2^2 = 0$.

Now any circle touching PB at P may be represented as family of point circle having center at P and

the line PB i.e. $(x - at_1t_2)^2 + (y - a(t_1 + t_2))^2 + l(x - t_2y + at_2^2) = 0$.

As this circle passes through $F(a, 0)$, hence $l = -a(1 + t_1^2)$.

Now the circle touching PB at P & passing through F is

$$(x - at_1t_2)^2 + (y - a(t_1 + t_2))^2 - a(1 + t_1^2)(x - t_2y + at_2^2) = 0$$

Substituting coordinates of A in L.H.S. of equation of circle gives

$(at_1^2 - at_1t_2)^2 + (2at_1 - a(t_1 + t_2))^2 - a(1 + t_1^2)(at_1^2 - 2at_1t_2 + at_2^2)$ or $(t_1 - t_2)^2(t_1^2 + 1 - (1 + t_1^2))$ which is zero, hence this circle passes through A .

Q.18

(i) Let P, Q & R be the vertices of a triangle formed by three tangents of $y^2 = 4ax$, then the coordinates of these points can be taken as $(at_1t_2, a(t_1 + t_2))$, $(at_2t_3, a(t_2 + t_3))$ & $(at_3t_1, a(t_3 + t_1))$. Also the focus is S(a, 0).

$$\text{Now } m_{PQ} = \frac{1}{t_2}, m_{PR} = \frac{1}{t_1}, m_{FQ} = \frac{t_2 + t_3}{t_2t_3 - 1} \text{ \& } m_{FR} = \frac{t_3 + t_1}{t_3t_1 - 1}.$$

$$\text{Let angle between PQ \& PR be } a, \text{ then } \tan a = \frac{\frac{1}{t_2} - \frac{1}{t_1}}{1 + \frac{1}{t_2} \frac{1}{t_1}} \text{ i.e. } \frac{t_1 - t_2}{t_1t_2 + 1}.$$

$$\text{Similarly let angle between FQ \& FR be } b, \text{ then } \tan b = \frac{\frac{t_3 + t_1}{t_3t_1 - 1} - \frac{t_2 + t_3}{t_2t_3 - 1}}{1 + \frac{t_3 + t_1}{t_3t_1 - 1} \frac{t_2 + t_3}{t_2t_3 - 1}} \text{ i.e. } \frac{t_2 - t_1}{t_1t_2 + 1}.$$

(Here take care to put slopes in same cyclic order to get correct angles)

Clearly a & b are supplementary angles, hence PQFR is a cyclic quadrilateral.

(ii) Altitude through P must be perpendicular to tangent QR, hence its slope will be $-t_3$.

$$\text{Equation of this altitude will be } y - a(t_1 + t_2) = -t_3(x - at_1t_2).$$

$$\text{Similarly altitude through Q will be } y - a(t_2 + t_3) = -t_1(x - at_2t_3).$$

Eliminating y between these two equations gives $x = -a$, hence orthocenter lies on directrics.

Q.19

Any circle touching the parabola at P(t) will also touch the tangent to parabola at P.

Now any circle touching the line $y = tx + at^2$ at P may be represented as family of point circle having

$$\text{center at P and the line PB i.e. } (x - at^2)^2 + (y - 2at)^2 + l(x - ty + at^2) = 0.$$

$$\text{As this circle passes through F(a, 0), hence } l = -a(1 + t^2).$$

Hence the circle touching the parabola at P & passing through F is

$$(x - at^2)^2 + (y - 2at)^2 - a(1 + t^2)(x - ty + at^2) = 0$$

Similarly the circle touching the parabola at Q & passing through F is

$$\left(x - \frac{a}{t^2}\right)^2 + \left(y + \frac{2a}{t}\right)^2 - a\left(1 + \frac{1}{t^2}\right)\left(x + \frac{1}{t}y + \frac{a}{t^2}\right) = 0, \text{ note that P \& Q are end points of a focal chord.}$$

The two equation of circles simplify to

$$x^2 + y^2 - a(3t^2 + 1)x + at(t^2 - 3)y + 3a^2t^2 = 0 \text{ \& } x^2 + y^2 - a\left(\frac{3}{t^2} + 1\right)x + \frac{a}{t}\left(3 - \frac{1}{t^2}\right)y + \frac{3a^2}{t^2} = 0.$$

$$\text{Now } g_1 = \frac{a(3t^2 + 1)}{2}, f_1 = -\frac{at(t^2 - 3)}{2}, c_1 = 3a^2t^2 \text{ \& } g_2 = \frac{a(3 + t^2)}{2t^2}, f_2 = -\frac{a(3t^2 - 1)}{2t^3}, c_2 = \frac{3a^2}{t^2} \text{ gives}$$

$$2g_1g_2 + 2f_1f_2 = 2 \frac{a(3t^2 + 1)}{2} \frac{a(3 + t^2)}{2t^2} + 2 \frac{at(t^2 - 3)}{2} \frac{a(3t^2 - 1)}{2t^3} \text{ or } 2g_1g_2 + 2f_1f_2 = a^2 \left[\frac{3t^4 + 3}{t^2} \right] = c_1 + c_2.$$

Hence the circles are orthogonal.

Q.20

Equation of line joining (1, 0) & (0, 2) is $2x + y = 2$.

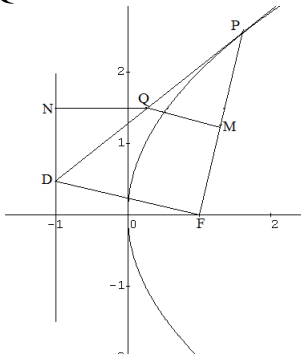
Now any curve having $xy = 0$ as pair of tangents and $2x + y - 2 = 0$ as chord of contact may be represented as $(2x + y - 2)^2 + lxy = 0$ or $4x^2 + (l + 4)xy + y^2 - 8x - 4y + 4 = 0$.

For this equation to represent a parabola $h^2 = ab \Rightarrow \left(\frac{l + 4}{2}\right)^2 = 4 \Rightarrow l = -8$ or 0 .

But for $l = 0$ the equation becomes $(2x + y - 2)^2 = 0$.

Hence required parabola is $4x^2 - 4xy + y^2 - 8x - 4y + 4 = 0$.

Q.21



Consider the parabola $y^2 = 4ax$.

Let P be $(at^2, 2at)$ & Q be (h, k) . Also equation of PQ will be

$$ty = x + at^2, \text{ hence } k = \frac{h + at^2}{t}.$$

Now slope of FP is $\frac{2t}{t^2 - 1}$, hence equation of QM will be

$$(y - k) = \frac{1 - t^2}{2t}(x - h).$$

Also QN will be parallel to x-axis thus its equation will be $y = k$.

Hence $QN = a + h$.

Now perpendicular distance of QM from F i.e. $FM = \frac{\left| k + \frac{1 - t^2}{2t}(a - h) \right|}{\sqrt{\left(\frac{1 - t^2}{2t}\right)^2 + 1}}$.

Therefore $FM = \frac{\left| \frac{h + at^2}{t} + \frac{1 - t^2}{2t}(a - h) \right|}{\sqrt{\left(\frac{1 - t^2}{2t}\right)^2 + 1}} \Rightarrow FM = \frac{(h + a)(1 + t^2)}{\sqrt{(1 - t^2)^2 + 4t^2}}$

Or $FM = h + a = QN$.

Q.22

We can get the solution by first consider a fixed parabola touching the coordinate axes and then rotating it by an angle α .

One such parabola is $x^2 - 2xy + y^2 - 2ax - 2ay + a^2 = 0$ which touches the coordinate axes at (the end points of latus rectum i.e. $(a, 0)$ & $(0, a)$). Equation of its latus rectum is $x + y = a$.

Rotating the parabola by an angle α transforms the equation of latus rectum into

$$x(\cos\alpha + \sin\alpha) + y(\cos\alpha - \sin\alpha) = a, \text{ which may be rearranged as}$$

$y = -\frac{\cos Q + \sin Q}{\cos Q - \sin Q}x + \frac{a}{\cos Q - \sin Q}$. Now let $\frac{\cos Q + \sin Q}{\cos Q - \sin Q} = -m$, then the equation reduces to

$$y = mx - \frac{a}{\sqrt{2}}\sqrt{1+m^2}, \text{ which is equation of tangent to the circle } x^2 + y^2 = \frac{a^2}{2}.$$

Q.23

Let vertices of the triangle be $(at_1^2, 2at_1), (at_2^2, 2at_2), (at_3^2, 2at_3)$.

Now sides joining these will be

$$y = \frac{2}{t_1 + t_2}x + \frac{2at_1t_2}{t_1 + t_2}, y = \frac{2}{t_2 + t_3}x + \frac{2at_2t_3}{t_2 + t_3}, y = \frac{2}{t_3 + t_1}x + \frac{2at_3t_1}{t_3 + t_1}.$$

Let the first line touch $x^2 = 4by$, then $\frac{2at_1t_2}{t_1 + t_2} = -b\left(\frac{2}{t_1 + t_2}\right)^2$ or $at_1t_2(t_1 + t_2) = -2b$.

Similarly if the second line is a tangent then $at_2t_3(t_2 + t_3) = -2b$.

Now from these two conditions we get $t_1 + t_2 + t_3 = 0$ & $t_2t_3t_1 = \frac{2b}{a}$.

Further $at_3t_1(t_3 + t_1) = a \cdot \frac{2b}{at_2} \cdot (-t_2) = -2b$, hence the third line also touch $x^2 = 4by$.

Q.24

If the triangle is equilateral, then its centroid will be same as circum center.

Let the vertices be $(at_1^2, 2at_1), (at_2^2, 2at_2), (at_3^2, 2at_3)$.

$$\text{Centroid will be } h = \frac{a(t_1^2 + t_2^2 + t_3^2)}{3} \text{ \& } k = \frac{2a(t_1 + t_2 + t_3)}{3}.$$

Now consider the circle $x^2 + y^2 - 2hx - 2ky + c = 0$ and put $(at^2, 2at)$ in this equation to get $a^2t^4 + 2a(2a - h)t^2 - 4akt + c = 0$.

Now if this is the circum circle of triangle PQR, then t_1, t_2, t_3 will be three of its roots.

Using relations in roots and coefficients we get

$$t_1 + t_2 + t_3 + t_4 = 0, t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{2(2a - h)}{a} \text{ \&}$$

$$t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 = \frac{4k}{a}.$$

$$\text{Also } h = \frac{a(t_1^2 + t_2^2 + t_3^2)}{3} \text{ \& } k = \frac{2a(t_1 + t_2 + t_3)}{3} \text{ gives } t_4 = -\frac{3k}{2a} \text{ \& } t_1t_2 + t_1t_3 + t_1t_4 = \frac{12ah - 9k^2}{8a^2}$$

Now from $t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{2(2a - h)}{a}$ we get

$$t_1t_2 + t_2t_3 + t_3t_1 + (t_1 + t_2 + t_3)t_4 = \frac{2(2a - h)}{a}.$$

Substituting the values gives $4ah - 9k^2 - 32a^2 = 0$.

Hence locus of centroid of triangle PQR is $9y^2 - 4ax + 32a^2 = 0$.

Q.25

Let extremities of the focal chord be $P(at^2, 2at), Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Point of intersection of tangents at P & Q will be $R\left(-a, 2a\left(t - \frac{1}{t}\right)\right)$.

Now area of triangle PQR will be

$$\frac{1}{2} \begin{vmatrix} 1 & at^2 & 2at \\ 1 & \frac{a}{t^2} & -\frac{2a}{t} \\ 1 & -a & a\left(t - \frac{1}{t}\right) \end{vmatrix} \text{ i.e. } \frac{a^2}{2} \left(t^2 + \frac{1}{t^2} + 2\right) \left|t + \frac{1}{t}\right|.$$

Similarly area of triangle OPQ will be

$$\frac{1}{2} \begin{vmatrix} 1 & at^2 & 2at \\ 1 & \frac{a}{t^2} & -\frac{2a}{t} \\ 1 & 0 & 0 \end{vmatrix} \text{ i.e. } a^2 \left|t + \frac{1}{t}\right|.$$

Now ratio of these two area will be $\frac{1}{2} \left(t^2 + \frac{1}{t^2}\right) + 1$.

Q.26

Let slope of the variable line be $\tan\theta$.

Now any point on this line at a distance r from $P(a, b)$ will be $(a + r\cos\theta, b + r\sin\theta)$.

These coordinates will satisfy $y^2 = 4cx$ for $r = PA$ & $r = PB$.

Hence $(b + r\sin\theta)^2 = 4c(a + r\cos\theta)$ i.e. $(\sin^2\theta)r^2 + (2b\sin\theta - 4c\cos\theta)r + b^2 - 4ac = 0$ will have PA & PB as roots. Now

$$PA + PB = \frac{4c\cos\theta - 2b\sin\theta}{\sin^2\theta} \text{ \& } PA \cdot PB = \frac{4ac - b^2}{\sin^2\theta}.$$

As given PA, PQ, PB are in H.P., hence $PQ = \frac{2 \cdot PA \cdot PB}{PA + PB} \Rightarrow PQ = \frac{2(4ac - b^2)}{4c\cos\theta - 2b\sin\theta}$.

Now let coordinate of Q be (x, y) , then

$$x = a + PQ\cos\theta \text{ \& } y = b + PQ\sin\theta \Rightarrow \cos\theta = \frac{x - a}{PQ} \text{ \& } \sin\theta = \frac{y - b}{PQ}.$$

Substituting these in the expression of PQ we get $PQ = \frac{2(4ac - b^2)}{4c(x - a) - 2b(y - b)} PQ$

$$\text{or } 2cx - by = 6ac - 2b^2.$$

Hence locus of Q is a fixed straight line.

Q.27

Foot of perpendicular from the focus F on tangent at P will lie on y-axis, hence let P, F & M be $(at^2, 2at), (a, 0) \text{ \& } (0, at)$.

Now area of triangle PFM will be

$$\frac{1}{2} \left\| \begin{vmatrix} 1 & at^2 & 2at \\ 1 & a & 0 \\ 1 & 0 & at \end{vmatrix} \right\| = \frac{a^2}{2} (t^3 + t).$$

Now range of t is 0 to 1.

Maximum area will be for $t = 1$ i.e. maximum area = a^2 .

Q.28

Let mid point of any such chord be $M(at^2, 2at)$.

Now using $T = S_1$, equation of chord of $x^2 + y^2 = 16a^2$ having mid point at M may be represented as $at^2x + 2aty = a^2t^4 + 4a^2t^2$.

As this chord is drawn through $(h, 0)$ hence substituting these coordinates in equation of chord we get $at^2x + 2aty = a^2t^4 + 4a^2t^2$.

Now the above equation gives three values of t , namely 0 & $\pm \sqrt{\frac{h-4a}{a^2}}$ out of which the later two values will be real & other than 0 only if $h > 4a$.

Also for M to be mid point of chord it must lie inside the circle hence

$$a^2t^4 + 4a^2t^2 - 16a^2 < 0 \text{ or } \left(\frac{4a-h}{a^2}\right)^2 + 4\left(\frac{4a-h}{a^2}\right) - 16 < 0, \text{ hence } h < (\sqrt{5}+1)2a.$$

Q.29

Reflection at a point P on any curved surface take place such that incident ray and reflected are reflections of each other in the normal to the curve at P .

Now $y = b$ meets $y^2 = 4ax$ at point $P\left(\frac{b^2}{4a}, b\right)$. Comparing this with $(at^2, 2at)$ gives $t = \frac{b}{2a}$.

Normal to the parabola at this point will be

$$\frac{b}{2a}x + y = 2a\frac{b}{2a} + a\left(\frac{b}{2a}\right)^3 \text{ or } 4abx + 8a^2y = 8a^2b + b^3.$$

Now slope of normal is $-\frac{b}{2a}$ and $y = b$ is parallel to x -axis so if q is the angle between the incident

ray and normal, then $\tan q = -\frac{b}{2a}$.

Reflected ray will make an angle $2q$ with $y = b$, hence slope of reflected ray will be

$$\tan 2q = \frac{2 \tan q}{1 - \tan^2 q} = \frac{4ab}{4a^2 - b^2}.$$

$$\text{Equation of the reflected ray : } y - b = \frac{4ab}{4a^2 - b^2} \left(x - \frac{b^2}{4a}\right) \text{ or } 4abx + (4a^2 - b^2)y = 4a^2b.$$

Clearly $(a, 0)$ satisfies this equation.

Q.30

If the triangle is equilateral, then its centroid will be same as circum center.

Let the vertices be $(at_1^2, 2at_1), (at_2^2, 2at_2), (at_3^2, 2at_3)$.

$$\text{Centroid will be } h = \frac{a(t_1^2 + t_2^2 + t_3^2)}{3} \text{ \& } k = \frac{2a(t_1 + t_2 + t_3)}{3}.$$

Now consider the circle $x^2 + y^2 - 2hx - 2ky + c = 0$ and put $(at^2, 2at)$ in this equation to get $a^2t^4 + 2a(2a - h)t^2 - 4akt + c = 0$.

Now if this is the circum circle of triangle PQR, then t_1, t_2, t_3 will be three of its roots.

Using relations in roots and coefficients we get

$$t_1 + t_2 + t_3 + t_4 = 0, t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{2(2a - h)}{a} \&$$

$$t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 = \frac{4k}{a}.$$

$$\text{Also } h = \frac{a(t_1^2 + t_2^2 + t_3^2)}{3} \& k = \frac{2a(t_1 + t_2 + t_3)}{3} \text{ gives } t_4 = -\frac{3k}{2a} \& t_1t_2 + t_1t_3 + t_1t_4 = \frac{12ah - 9k^2}{8a^2}$$

Now from $t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{2(2a - h)}{a}$ we get

$$t_1t_2 + t_2t_3 + t_3t_1 + (t_1 + t_2 + t_3)t_4 = \frac{2(2a - h)}{a}.$$

Substituting the values gives $4ah - 9k^2 - 32a^2 = 0$.

Hence locus of centroid of triangle PQR is $9y^2 - 4ax + 32a^2 = 0$.

Q.31

Given data implies that point of intersection of two normal lies on the parabola.

Let a normal be drawn at $P(\lambda)$, then its equation will be $|x + y = 2a\lambda + a\lambda^3$.

If it passes through $(at^2, 2at)$, then $|at^2 + 2at = 2a\lambda + a\lambda^3$.

$$\Rightarrow a\lambda(t^2 - \lambda^2) + 2a(t - \lambda) = 0 \text{ or } \lambda^2 + \lambda t + a = 0.$$

Q.32

Let the points on parabola be $A(at_1^2, 2at_1), B(at_2^2, 2at_2) \& C(at_3^2, 2at_3)$.

Points of intersection of tangents at these points will be

$P(at_1t_2, a(t_1 + t_2)), Q(at_2t_3, a(t_2 + t_3)) \& R(at_3t_1, a(t_3 + t_1))$.

$$\text{Now Area of } \triangle ABC = \frac{1}{2} \begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^2 & 2at_3 \end{vmatrix} \& \text{Area of } \triangle PQR = \frac{1}{2} \begin{vmatrix} 1 & at_2t_3 & a(t_2 + t_3) \\ 1 & at_3t_1 & a(t_3 + t_1) \\ 1 & at_1t_2 & a(t_1 + t_2) \end{vmatrix}.$$

Now take the second determinant,

(i) Subtract $a(t_1 + t_2 + t_3)$ from third column to get

$$\text{Area of } \triangle PQR = \frac{1}{2} \begin{vmatrix} 1 & at_2t_3 & -at_1 \\ 1 & at_3t_1 & -at_2 \\ 1 & at_1t_2 & -at_3 \end{vmatrix}.$$

(ii) Multiply first row by t_1 , second by t_2 & third by t_3 and take $t_1t_2t_3$ common from second column. Also take a common from second column and multiply $2a$ to first column to get

$$\text{Area of DPQR} = \frac{1}{4} \begin{vmatrix} 2at_1 & 1 & at_1^2 \\ 2at_2 & 1 & at_2^2 \\ 2at_3 & 1 & at_3^3 \end{vmatrix}.$$

(iii) Now interchange first column with second and then second with third to get

$$\text{Area of DPQR} = \frac{1}{4} \begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^3 & 2at_3 \end{vmatrix} = \frac{1}{2} \text{Area of DABC}.$$

Q.33

Let the points be $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$, where as given $t_2 = 2t_1$.

Now point of intersection of normal at P & Q will be

$$x = a(t_1^2 + t_2^2 + t_1t_2 + 2) \text{ \& } y = -at_1t_2(t_1 + t_2)$$

$$\text{Now } t_2 = 2t_1 \text{ } \Rightarrow x - 2a = 7at_1^2 \text{ \& } y = -6at_1^3.$$

$$\text{Eliminating } t_1 \text{ gives } 36(x - 2a)^3 = 243ay^2.$$

Q.34

Let the points be $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$, where as given $t_1t_2 = -1$.

Now point of intersection of normal at P & Q will be

$$x = a(t_1^2 + t_2^2 + t_1t_2 + 2) \text{ \& } y = -at_1t_2(t_1 + t_2).$$

$$\text{Now } t_1t_2 = -1 \text{ gives } x = a(t_1^2 + t_2^2 + 1) \text{ \& } y = a(t_1 + t_2).$$

$$\text{Eliminating } t_1 \text{ \& } t_2 \text{ gives } a(x - 3a) = y^2.$$

Q.35

Let the points be $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$, where as given $t_1t_2 = 2$.

$$\text{Now mid point of P \& Q will be } x = \frac{a(t_1^2 + t_2^2)}{2} \text{ \& } y = a(t_1 + t_2).$$

$$\text{Now } t_1t_2 = 2 \text{ gives } 2x + 4a = a(t_1 + t_2)^2 \text{ \& } y = a(t_1 + t_2).$$

$$\text{Eliminating } t_1 \text{ \& } t_2 \text{ gives } 2a(x + 2a) = y^2.$$