EXERCISE 1(A)

INDEFINITE INTEGRATION

1 $\int \sin^2(x/2)\,dx$ equals-

(A) $\frac{1}{2}$ (x + sin x) + c (B) $\frac{1}{2}$(x + cos x) + c

(C) $\frac{1}{2}$ (x – sin x) + c (D) None of these

Sol. Here I = $\int \frac{1-\cos x}{2}\,dx = \frac{1}{2}$ (x – sin x) + c **Ans.[C]**

2 $\int \cot^2 x\,dx$ equals -

(A) – sec x + x + c (B) – cot x – x + c

(C) – sin x + x + c (D) None of these

Sol. $\int$ (cosec^2 x – 1) dx = – cot x – x + c **Ans. [B]**

3 $\int \frac{5x+7}{x}\,dx$ equals-

(A) 5x + 7 log x (B) 7x + 5 log x +c

(C) 5x + 7 log x + c (D) None of these

Sol. $\int \frac{5x+7}{x}\,dx = \int\left(\frac{5x}{x}+\frac{7}{x}\right)dx$

$= \int 5\,dx + \int \frac{7}{x} = 5\int 1\,dx + 7\int \frac{1}{x}\,dx = 5x + 7\log x + c$ **Ans.[C]**

4 $\int\left(x-\frac{1}{x}\right)^3 dx$, (x > 0) equals-

(A) $\frac{x^3}{3} - \frac{3}{2}x^2 + 3\log x + \frac{1}{2x^2} + c$ (B) $\frac{x^4}{3} - \frac{3}{2}x^2 + 3\log x + \frac{1}{2x^2} + c$

(C) $\frac{x^4}{4} + 3\log x + \frac{1}{2x^2} + c$ (D) None of these

Sol. $\int\left(x-\frac{1}{x}\right)^3 dx$

$= \int\left(x^3 - 3x^2.\frac{1}{x} + 3x.\frac{1}{x^2} - \frac{1}{x^3}\right)$Edx

[$\because (a-b)^3 = (a^3 - 3a^2b + 3ab^2 - b^3)$]

$= \int\left(x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}\right)dx$

$= \int x^3\,dx - 3\int x\,dx + 3\int \frac{1}{x}\,dx - \int \frac{1}{x^3}\,dx = \frac{x^{3+1}}{3+1} - 3.\ \frac{x^{1+1}}{1+1} + 3\log x - \frac{x^{-3+1}}{-3+1} + c$

$= \frac{x^4}{4} - \frac{3}{2}x^2 + 3\log x + \frac{1}{2x^2} + c$ **Ans.[B]**

5 The value of $\int\left(\frac{6}{1+x^2}+10^x\right)dx$ is -

(A) $6\tan^{-1}x + 10^x \log_e 10 + c$

(B) $6\tan^{-1}x + \frac{10^x}{\log_e 10} + c$

(C) $3\tan^{-1}x + \frac{10^x}{\log_e 10} + c$

(D) None of these

Sol. $\int\left(\frac{6}{1+x^2}+10^x\right)dx$

$= 6\int\frac{1}{1+x^2}dx + \int 10^x\,dx = 6\tan^{-1}x + \frac{10^x}{\log_e 10} + C$ **Ans.[B]**

6 $\int(\tan x + \cot x)^2\,dx$ is equal to-

(A) $\tan x - \cot x + c$

(B) $\tan x + \cot x + c$

(C) $\cot x - \tan x + c$

(D) None of these

Sol. $I = \int(\tan^2 x + \cot^2 x + 2)\,dx$

$= \int(\sec^2 x + \operatorname{cosec}^2 x)\,dx$

$= \tan x - \cot x + c$ **Ans. [A]**

7 $\int \sin 2x \sin 3x\,dx$ equals-

(A) $\frac{1}{2}(\sin x - \sin 5x) + c$

(B) $\frac{1}{10}(\sin x - \sin 5x) + c$

(C) $\frac{1}{10}(5\sin x - \sin 5x) + c$

(D) None of these

Sol. $I = \frac{1}{2}\int[\cos(-x) - \cos 5x]\,dx$

$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right] + c$

$= \frac{1}{10}[5\sin x - \sin 5x] + c$ **Ans. [C]**

8 $\int\frac{x^2}{x^2-1}dx$ equals-

(A) $x + \log\sqrt{\frac{x-1}{x+1}} + c$

(B) $x + \log\sqrt{\frac{x+1}{x-1}} + c$

(C) $x + \log\left(\frac{x-1}{x+1}\right) + c$

(D) $x + \log\left(\frac{x+1}{x-1}\right) + c$

Sol. $\int\frac{x^2-1+1}{x^2-1}dx$

$= \int\left(1+\frac{1}{x^2-1}\right)dx = x + \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + c$

$= x + \log\sqrt{\frac{x-1}{x+1}} + c$ **Ans.[A]**

9 $\int \frac{x^5}{\sqrt{1+x^3}}$ dx equals-

(A) $\frac{2}{9}(x^3-2)\sqrt{1+x^3}+c$ (B) $\frac{2}{9}(x^3+2)\sqrt{1+x^3}+c$

(C) $(x^3+2)\sqrt{1+x^3}+c$ (D) None of these

Sol. Put $1+x^3=t^2 \Rightarrow 3x^2\,dx = 2\,t\,dt$

$$\therefore I = \int \frac{x^3}{\sqrt{1+x^3}}(x^2\,dx) = \frac{2}{3}\int (t^2-1)\,dt$$

$$= \frac{2}{3}\left[\frac{t^3}{3}-t\right]+c$$

$$= \frac{2}{3}\left[\frac{1}{3}(1+x^3)^{3/2}-\sqrt{1+x^3}\right]+c$$

$$= \frac{2}{9}\sqrt{1+x^3}\,(1+x^3-3)+c$$

$$= \frac{2}{9}(x^3-2)\sqrt{1+x^3}+c$$ **Ans. [A]**

10 $\int \frac{1}{x\log x}$ dx is equal to-

(A) log (x log x) + c (B) log (log x + x) + c

(C) log x + c (D) log (log x) + c

Sol. $\int \frac{1}{x\log x}dx = \int \frac{1}{x}\cdot\frac{1}{\log x}dx$

put log x = t, $\frac{1}{x}dx = dt$

$$\therefore \int \frac{1}{x}\cdot\frac{1}{\log x}dx = \int \frac{1}{t}\,dt$$

$$\therefore \int \frac{1}{t}dt = \log t + c = \log(\log x) + c$$

(putting the value of t = log x) **Ans.[D]**

11 $\int \sec^2 x \cos(\tan x)\,dx$ equals-

(A) sin (cos x) + c (B) sin (tan x) + c

(C) cosec (tan x) + c (D) None of these

Sol. Let tan x = t, then $\sec^2 x\,dx = dt$

$$\therefore I = \int \cos t\,dt = \sin t + c$$

$$= \sin(\tan x) + c$$ **Ans.[B]**

12 $\int \tan^n x \sec^2 x\,dx$ equals-

(A) $\frac{\tan^{n-1}x}{n-1}+c$ (B) $\frac{\tan^{n-1}x}{n+1}+c$

(C) $\tan^{n+1}x + c$ (D) None of these

Sol. $\int \tan^n x \sec^2 x\,dx$

putting tan x = t, $\sec^2 x\,dx = dt$

$\int \tan^n x \sec^2 x\, dx = \int t^n\, dt = \frac{\tan^{n+1}}{n+1} + c$

$= \frac{(\tan x)^{n+1}}{n+1} + c$ **Ans.[B]**

13 $\int \frac{\sin 2x}{1+\cos^4 x} dx$ is equal to-

(A) $\cos^{-1}(\cos^2 x) + c$ (B) $\sin^{-1}(\cos^2 x) + c$

(C) $\cot^{-1}(\cos^2 x) + c$ (D) None of these

Sol. Here differential coefficient of

$\cos^2 x$ is $-\sin 2x$

Let $\cos^2 x = t$

$\therefore 2\cos x(-\sin x)\, dx = dt$

or $\sin 2x\, dx = -dt$

$\therefore \int \frac{\sin 2x}{1+\cos^4 x} dx = \int \frac{-dt}{1+t^2}$

$= \cot^{-1} t + c$

$= \cot^{-1}(\cos^2 x) + c$ **Ans.[C]**

14 $\int \frac{be^x}{\sqrt{a+be^x}} dx$ equals-

(A) $\frac{2}{b}\sqrt{a+be^x} + c$ (B) $\frac{1}{b}\cdot\sqrt{a+be^x} + c$

(C) $2\sqrt{a+be^x} + c$ (D) None of these

Sol. $\int \frac{be^x}{\sqrt{a+be^x}} dx$, putting $a + be^x = t$

$be^x\, dx = dt$

$\therefore \int \frac{be^x}{\sqrt{a+be^x}} dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + c$

$= 2\sqrt{a+be^x} + c$ **Ans.[C]**

15 $\int \sqrt{\frac{1+\cos x}{1-\cos x}}\, dx$ equals-

(A) $\log \cos\left(\frac{x}{2}\right) + c$ (B) $2\log \sin\left(\frac{x}{2}\right) + c$

(C) $2\log \sec\left(\frac{x}{2}\right) + c$ (D) None of these

Sol. $I = \int \sqrt{\frac{1+\cos x}{1-\cos x}}\, dx$

$= \int \sqrt{\frac{2\cos^2(x/2)}{2\sin^2(x/2)}}\, dx$

$= \int \cot\left(\frac{x}{2}\right) dx$

$= 2\log \sin\left(\frac{x}{2}\right) + c$ **Ans.[B]**

16 $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$ equals-

(A) $2\sqrt{\sec x} + c$ (B) $2\sqrt{\tan x} + c$

(C) $2/\sqrt{\tan x} + c$ (D) $2/\sqrt{\sec x} + c$

Sol. $I = \int \frac{\sqrt{\tan x}}{\tan x} \sec^2 x \, dx$

$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx = 2\sqrt{\tan x} + c$ **Ans. [B]**

17 $\int \sin^5 x. \cos^3 x \, dx$ is equal to-

(A) $\frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$ (B) $\frac{\cos^6 x}{6} - \frac{\cos^8 x}{8} + c$

(C) $\frac{\cos^6 x}{6} - \frac{\sin^8 x}{8} + c$ (D) None of these

Sol. $\int \sin^5 x . \cos^3 x \, dx$

Assumed that $\sin x = t$

$\therefore \cos x \, dx = dt$

$= \int t^5(1 - t^2) \, dt = \int (t^5 - t^7) \, dt$

$= \frac{t^6}{6} - \frac{t^8}{8} + c = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$ **Ans.[A]**

18 $\int \frac{x^2}{1+x^6} dx$ is equal to-

(A) $\tan^{-1}x^3 + c$ (B) $\tan^{-1}x^2 + c$

(C) $\frac{1}{3}\tan^{-1}x^3 + c$ (D) $3\tan^{-1}x^3 + c$

Sol. Put $x^3 = t \Rightarrow x^2 \, dx = \frac{1}{3} dt$

$\therefore I = \frac{1}{3} \int \frac{dt}{1+t^2} = \frac{1}{3} \tan^{-1} x^3 + c$ **Ans. [C]**

19 $\int \sqrt{\frac{1+x}{1-x}} dx$ equals-

(A) $\sin^{-1} x + \sqrt{1-x^2} + c$ (B) $\sin^{-1} x + \sqrt{x^2-1} + c$

(C) $\sin^{-1} x - \sqrt{1-x^2} + c$ (D) $\sin^{-1} x - \sqrt{x^2-1} + c$

Sol $I = \int \sqrt{\frac{1+x}{1-x}} dx$

$= \int \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2} \int \frac{-2x \, dx}{\sqrt{1-x^2}}$

$= \sin^{-1} x - \sqrt{1-x^2} + c$ **Ans. [C]**

20 The primitive of log x will be-

(A) $x \log (e + x) + c$

(B) $x \log\left(\frac{e}{x}\right) + c$

(C) $x \log\left(\frac{x}{e}\right) + c$

(D) $x \log (ex) + c$

Sol. $\int \log x \, dx = \int \log x.1 \, dx$

[Integrating by parts, taking log x as first part and 1 as second part]

$= (\log x).x - \int \left\{\frac{d(\log x)}{dx}\right\}. x \, dx$

$= x \log x - \int \frac{1}{x} .x \, dx = (x \log x - x) + c$

$= x (\log x - 1) + c = \log\left(\frac{x}{e}\right) + c$ **Ans. [C]**

21 $\int x \tan^{-1} x$ is equal to-

(A) $\frac{1}{2}(x^2 + 1) \tan^{-1} x - x + c$

(B) $\frac{1}{2}(x^2 + 1) \tan^{-1} x + x + c$

(C) $\frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2} x + c$

(D) $\frac{1}{2}(x^2 - 1) \tan^{-1} x - \frac{1}{2} x + c$

Sol. Integrating by parts taking x as second part

$I = \frac{x^2}{2} \tan^{-1} x - \int \frac{1}{1+x^2} . \frac{x^2}{2} dx$

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2}\left(1 - \frac{1}{1-x^2}\right) dx$

$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$

$= \frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2} x + c$ **Ans. [C]**

22 $\int \sin (\log x) \, dx$ equals-

(A) $\frac{x}{\sqrt{2}} \sin (\log x + \frac{\pi}{8}) + c$

(B) $\frac{x}{\sqrt{2}} \sin (\log x - \frac{\pi}{4}) + c$

(C) $\frac{x}{\sqrt{2}} \cos (\log x - \frac{\pi}{4}) + c$

(D) None of these

Sol. $\int \sin (\log x) \, dx$, assumed that $x = e^t$

$\therefore dx = e^t \, dt$

$= \int \sin t.e^t.dt$

$= \frac{e^t}{\sqrt{1+1}} \sin(t - \tan^{-1} 1) + c$

$\Rightarrow \int \sin (\log x) \, dx$

$= \frac{x}{\sqrt{2}} \sin (\log x - \frac{\pi}{4}) + c$ **Ans. [B]**

23 $\int \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x+a}}\,dx$ equals-

(A) $\sqrt{x^2+ax} - 2\sqrt{ax+a^2} - a\cosh^{-1}\left(\sqrt{\frac{x+a}{a}}\right) + c$

(B) $\sqrt{x^2+ax} + \sqrt{ax+a^2} - a\cosh^{-1} + \left(\sqrt{\frac{x+a}{a}}\right) c$

(C) $\sqrt{x^2+ax} - 2\sqrt{ax+a^2} + a\cosh^{-1}\left(\sqrt{\frac{x+a}{a}}\right) + c$

(D) None of these

Sol. Let $x = a\tan^2\theta \Rightarrow dx = 2a\tan\theta\sec^2\theta\,d\theta$

$\therefore I = \int \frac{\sqrt{a}(\tan\theta - 1).2a\tan\theta\sec^2\theta}{\sqrt{a}\sec\theta}\,d\theta$

$= 2a\,[\int \tan^2\theta\sec\theta\,d\theta - \int \sec\theta\tan\theta\,d\theta\,]$

$= 2a\,[\int \sqrt{\sec^2\theta - 1}\,\tan\theta\sec\theta\,d\theta - \sec\theta] = 2a\int\sqrt{t^2-1}\,dt - 2a\sec\theta + c$ [Where $\sec\theta = t$]

$= 2a\left[\frac{t}{2}\sqrt{t^2-1} - \frac{1}{2}\cosh^{-1}(t)\right] - 2a\sqrt{\frac{a+x}{a}} + c$

$= a\sqrt{\frac{x+a}{a}.\frac{x}{a}} - a\cosh^{-1}\left(\sqrt{\frac{x+a}{a}}\right) - 2\sqrt{ax+a^2} + c$

$= \sqrt{x^2+ax} - 2\sqrt{ax+a^2} - a\cosh^{-1}\left(\sqrt{\frac{x+a}{a}}\right) + c$

Ans. [A]

24 $\int x^3(\log x)^2\,dx$ equals-

(A) $\frac{1}{32}x^4\,[8(\log x)^2 - 4\log x + 1] + c$

(B) $\frac{1}{32}x^4\,[8(\log x)^2 - 4\log x - 1] + c$

(C) $\frac{1}{32}x^4\,[8(\log x)^2 + 4\log x + 1] + c$

(D) None of these

Sol. Integrating by parts taking x^3 as second part

$I = \frac{1}{4}x^4(\log x)^2 - \frac{1}{2}\int x^3\log x\,dx$

$= \frac{1}{4}x^4(\log x)^2 - \frac{1}{2}\left(\frac{1}{4}x^4\log x - \frac{1}{16}x^4\right) + c$

$= \frac{1}{32}x^4\,[8(\log x)^2 - 4\log x + 1] + c$ **Ans. [A]**

25 The value of $\int x\sec x\tan x\,dx$ is-

(A) $x\sec x + \log(\sec x + \tan x) + c$

(B) $x\sec x - \log(\sec x - \tan x) + c$

(C) $x\sec x + \log(\sec x - \tan x) + c$

(D) None of the above

Sol. $\int x.(\sec x\tan x)\,dx$

$= (x.\sec x) - \int (1.\sec x)\,dx$

(Integrating by parts, taking x as first function)

$= x \sec x - \log(\sec x + \tan x) + c$

$= x \sec x - \log\left\{(\sec x + \tan x)\dfrac{\sec x - \tan x}{\sec x - \tan x}\right\} + c$

$= x \sec x - \log\left(\dfrac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right) + c$

$= x \sec x + \log(\sec x - \tan x) + c$ **Ans. [C]**

26 $\int \dfrac{\sin^{-1}\sqrt{x}}{\sqrt{1-x}}\,dx$ equals-

(A) $2[\sqrt{x} - \sqrt{1-x}\,\sin^{-1}\sqrt{x}\,] + c$
(B) $2[\sqrt{x} + \sqrt{1-x}\,\sin^{-1}\sqrt{x}\,] + c$
(C) $[\sqrt{x} - \sqrt{1-x}\,\sin^{-1}\sqrt{x}\,] + c$
(D) None of these

Sol. Let $x = \sin^2 t$, then

$dx = 2 \sin t \cos t\, dt$

$\therefore I = \int \dfrac{t}{\cos t} . 2 \sin t \cos t\, dt$

$= 2 \int t \sin t\, dt$

$= 2[-t \cos t + \sin t] + c \quad = 2[\sqrt{x} - \sqrt{1-x}\,\sin^{-1}\sqrt{x}\,] + c$ **Ans. [A]**

27 $\int e^x \dfrac{x-1}{(x+1)^3}\,dx$ equals-

(A) $-\dfrac{e^x}{x+1} + c$
(B) $\dfrac{e^x}{x+1} + c$
(C) $\dfrac{e^x}{(x+1)^2} + c$
(D) $-\dfrac{e^x}{(x+1)^2} + c$

Sol. $I = \int e^x \left[\dfrac{x+1-2}{(x+1)^3}\right] dx$

$= \int e^x \left(\dfrac{1}{(x+1)^2} - \dfrac{2}{(x+1)^3}\right) dx$

Thus the given integral is of the form

$= \int e^x \{f(x) + f'(x)\}\,dx$

$\therefore I = e^x f(x) = \dfrac{e^x}{(x+1)^2} + c$ **Ans.[C]**

28 $\int \sec^3\theta\, d\theta$ is equal to-

(A) $\dfrac{1}{2}[\tan\theta \sec\theta + \log(\tan\theta + \sec\theta)] + c$

(B) $\dfrac{1}{2}\tan\theta \sec\theta + \log(\tan\theta + \sec\theta) + c$

(C) $\frac{1}{2}$ [tan θ sec θ – log (tanθ + secθ)] + c

(D) None of these

Sol. $I = \int$ sec. sec^{2}θ. dθ

$= \int \sqrt{\tan^2\theta + 1}\sec^2\theta\, d\theta$

$= \int \sqrt{t^2+1}\, dt$, where t = tan θ

$= \frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log\left(t + \sqrt{t^2+1}\right) + c$

$= \frac{1}{2}$ [tanθ secθ + log (tanθ + secθ)] + c

Ans. [A]

29 $\int \frac{\cos x + x\sin x}{x(x+\cos x)}dx$ is equal to-

(A) log {x (x + cos x)}+ c

(B) $\log\left(\frac{x}{x+\cos x}\right) + c$

(C) $\log\left(\frac{x+\cos x}{x+\cos x}\right) + c$

(D) None of these

Sol. $I = \int \frac{(x+\cos x) - x + x\sin x}{x(x+\cos x)}dx$

$= \int \frac{1}{x}dx - \int \frac{1-\sin x}{x+\cos x}dx$

= log x – log (x + cos x) + c

$= \log\left(\frac{x}{x+\cos x}\right) + c$ **Ans. [B]**

30 $\int \sqrt{\sec x - 1}\, dx$ is equal to-

(A) 2 sin^{-1} $(\sqrt{2}\cos x/2) + c$

(B) –2 sinh^{-1} $(\sqrt{2}\cos x/2) + c$

(C) –2 cosh^{-1} $(\sqrt{2}\cos x/2) + c$

(D) None of these

Sol. $I = \int \sqrt{\frac{1-\cos x}{\cos x}}dx$

$= \int \frac{\sqrt{2}\sin x/2}{\sqrt{2\cos^2 x/2 - 1}}dx$

$= -2\int \frac{dt}{\sqrt{t^2-1}}$ where t = $\sqrt{2}$ cos x/2

= –2 cosh^{-1} t + c

= – 2 cosh^{-1} + ($\sqrt{2}$ cos x/2) +c **Ans. [C]**

31 $\int \frac{x^2+1}{(x-1)(x-2)}dx$ equals-

(A) $\log\left[\frac{(x-2)^5}{(x-1)^2}\right] + c$

(B) $x + \log\left[\frac{(x-2)^5}{(x-1)^2}\right] + c$

(C) $x + \log\left[\frac{(x-1)^5}{(x-2)^5}\right] + c$ (D) None of these

Sol. Here since the highest powers of x in Num^r and Den^r are equal and coefficients of x^2 are also equal,

therefore $\frac{x^2+1}{(x-1)(x-2)} \equiv 1 + \frac{A}{x-1} + \frac{B}{x-2}$

On solving we get A = – 2, B = 5

Thus $\frac{x^2+1}{(x-1)(x-2)} \equiv 1 - \frac{2}{x-1} + \frac{5}{x-2}$

The above method is used to obtain the value of constant corresponding to non repeated linear factor in the Den^r.

Now $I = \left(1 - \frac{2}{x-1} + \frac{5}{x-2}\right)dx$

$= x - 2\log(x-1) + 5\log(x-2) + c$

$= x + \log\left[\frac{(x-2)^5}{(x-1)^2}\right] + c$ **Ans.[B]**

32 The value of $\int \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ is-

(A) $\frac{1}{b^2-a^2}\left[b\tan^{-1}\frac{x}{b} - a\tan^{-1}\frac{x}{a}\right] + c$ (B) $\frac{1}{b^2-a^2}\left[a\tan^{-1}\frac{x}{b} - b\tan^{-1}\frac{x}{a}\right] + c$

(C) $\frac{1}{b^2-a^2}\left[b\tan^{-1}\frac{x}{b} + a\tan^{-1}\frac{x}{a}\right] + c$ (D) None of these

Sol. Putting $x^2 = y$ in integrand, we obtain

$\frac{y}{(y+a^2)(y+b^2)} = \frac{1}{b^2-a^2}\left[\frac{b^2}{y+b^2} - \frac{a^2}{y+a^2}\right]$

$\therefore I = \frac{1}{b^2-a^2} \cdot \left[\int \frac{b^2}{x^2+b^2}dx - \int \frac{a^2}{x^2+a^2}dx\right]$

$= \frac{1}{b^2-a^2}\left[b\tan^{-1}\frac{x}{b} - a\tan^{-1}\frac{x}{a}\right] + c$

Ans.[A]

33 $\int \frac{dx}{3x^2+2x+1}$ equals-

(A) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c$ (B) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c$

(C) $\frac{1}{\sqrt{2}}\cot^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c$ (D) None of these

Sol. $I = \frac{1}{3}\frac{dx}{x^2 + \frac{2}{3}x + \frac{1}{3}}$

$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \frac{2}{9}}$

$$= \frac{1}{3} \times \frac{3}{\sqrt{2}} \tan^{-1} + \left(\frac{x + \left(\frac{1}{3}\right)}{\sqrt{2}/3} \right) c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + c$$ **Ans.[A]**

34 $\int \sqrt{1 + x - 2x^2}\, dx$ equals-

(A) $\frac{1}{8}(4x - 1)\sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32} \sin^{-1}\left(\frac{4x-1}{3}\right) + c$

(B) $\frac{1}{8}(4x + 1)\sqrt{1 + x - 2x^2} - \frac{9\sqrt{2}}{32} \sin^{-1}\left(\frac{4x-1}{3}\right) + c$

(C) $\frac{1}{8}(4x - 1)\sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32} \cos^{-1}\left(\frac{4x-1}{3}\right) + c$

(D) None of these

Sol. $I = \sqrt{2} \int \sqrt{\frac{1}{2} - \left(x^2 - \frac{x}{2}\right)}\, dx$

$$= \sqrt{2} \int \sqrt{\left\{\frac{9}{16} - \left(x - \frac{1}{4}\right)^2\right\}}\, dx$$

$$= \sqrt{2} \left[\frac{1}{2}\left(x - \frac{1}{4}\right) \sqrt{\left\{\frac{9}{16} - \left(x - \frac{1}{4}\right)^2\right\}} \right.$$

$$\left. + \frac{9}{32} \sin^{-1}\left\{\frac{4}{3}\left(x - \frac{1}{4}\right)\right\} \right] + c$$

$$= \frac{1}{8}(4x - 1)\sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32} \sin^{-1}\left(\frac{4x-1}{3}\right) + c$$

Ans. [A]

35 $\int \frac{dx}{\sqrt{3 - 5x - x^2}}$ equals-

(A) $\sin^{-1}\left(\frac{2x+5}{\sqrt{37}}\right) + c$

(B) $\cos^{-1}\left(\frac{2x+5}{\sqrt{37}}\right) + c$

(C) $\sin^{-1}(2x + 5) + c$

(D) None of these

Sol. $I = \int \frac{dx}{\sqrt{\frac{37}{4} - \left(x + \frac{5}{2}\right)^2}}$

$$= \sin^{-1}\left(\frac{x + 5/2}{\sqrt{37}/2}\right) + c = \sin^{-1}\left(\frac{2x+5}{\sqrt{37}}\right) + c$$ **Ans. [A]**

36 $\int \sqrt{e^{2x} - 1}\, dx$ is equal to-

(A) $\sqrt{e^{2x} - 1} + \sec^{-1} e^{2x} + c$

(B) $\sqrt{e^{2x} - 1} - \sec^{-1} e^{2x} + c$

(C) $\sqrt{e^{2x} - 1} - \sec^{-1} e^{x} + c$

(D) None of these

Sol. $\int \frac{e^{2x}-1}{\sqrt{e^{2x}-1}}\,dx$

$= \frac{1}{2}\int \frac{2e^{2x}}{\sqrt{e^{2x}-1}}\,dx - \int \frac{e^{x}}{e^{x}\sqrt{e^{2x}-1}}\,dx$

$= \sqrt{e^{2x}-1} - \sec^{-1} e^{x} + c$ **Ans.[C]**

37 $\int \sqrt{\frac{e^{x}+a}{e^{x}-a}}\,dx$ is equal to-

(A) $\cos h^{-1}\left(\frac{e^{x}}{a}\right) + \sec^{-1}\left(\frac{e^{x}}{a}\right) + c$

(B) $\sin h^{-1}\left(\frac{e^{x}}{a}\right) + \sec^{-1}\left(\frac{e^{x}}{a}\right) + c$

(C) $\tan h^{-1}\left(\frac{e^{x}}{a}\right) + \cos^{-1}\left(\frac{e^{x}}{a}\right) + c$

(D) None of these

Sol. $\int \frac{e^{x}+a}{\sqrt{e^{2x}-a^{2}}}\,dx$

$= \int \frac{e^{x}}{\sqrt{e^{2x}-a^{2}}}\,dx + a\int \frac{e^{x}}{e^{x}\sqrt{e^{2x}-a^{2}}}\,dx$

$= \cosh^{-1}\left(\frac{e^{x}}{a}\right) + \sec^{-1}\left(\frac{e^{x}}{a}\right) + c$ **Ans.[A]**

38 $\int \frac{dx}{4\sin^{2}x + 4\sin x\cos x + 5\cos^{2}x}$ is equal to-

(A) $\tan^{-1}\left(\tan x + \frac{1}{2}\right) + c$

(B) $\frac{1}{4}\tan^{-1}\left(\tan x + \frac{1}{2}\right) + c$

(C) $4\tan^{-1}\left(\tan x + \frac{1}{2}\right) + c$

(D) None of these

Sol. After dividing by $\cos^{2} x$ to numerator and denominator of integration

$I = \int \frac{\sec^{2}x\,dx}{4\tan^{2}x + 4\tan x + 5}$

$= \int \frac{\sec^{2}x\,dx}{(2\tan x + 1)^{2} + 4}$

$= \frac{1}{2.2}\tan^{-1}\left(\frac{2\tan x + 1}{2}\right) + c$ **Ans. [B]**

39 $\int \left(\frac{1-x}{1+x}\right)^{2} dx$ is equal to-

(A) $x - 4\log(x+1) + \frac{4}{x+1} + c$

(B) $x - \log(x+1) + \frac{4}{x+1} + c$

(C) $x - 4\log(x+1) - \frac{4}{x+1} + c$

(D) $x + \log(x+1) - \frac{4}{x+1} + c$

Sol. $\int \frac{[2-(x+1)]^{2}}{(x+1)^{2}}\,dx$

$$= \int \left[\frac{4}{(x+1)^2} - \frac{4}{x+1} + 1 \right] dx$$

$$= -\frac{4}{x+1} - 4 \log (x + 1) + x + c$$ **Ans. [C]**

40 $\int \frac{e^x}{e^{2x} + 5e^x + 6}$ equals-

(A) $\log \left(\frac{e^x + 3}{e^x + 2} \right) + c$

(B) $\log \left(\frac{e^x + 2}{e^x + 3} \right) + c$

(C) $\frac{1}{2} \log \left(\frac{e^x + 2}{e^x + 3} \right) + c$

(D) None of these

Sol. Put $e^X = t \Rightarrow e^X\, dx = dt$

$$\therefore I = \int \frac{dt}{t^2 + 5t + 6} = \int \frac{dt}{(1+2)(t+3)}$$

$$= \int \left(\frac{1}{t+2} - \frac{1}{t+3} \right) dt$$

$$= \log\left(\frac{t+2}{t+3} \right) + c = \log\left(\frac{e^x + 2}{e^x + 3} \right) + c$$

Ans. [B]

41 $\int \frac{dx}{x + \sqrt{x}}$ equals-

(A) $2\log(\sqrt{x} - 1) + c$

(B) $2\log(\sqrt{x} + 1) + c$

(C) $\tan^{-1} x + c$

(D) None of these

Sol. $I = \int \frac{dx}{x + \sqrt{x}}$

$$= \int \frac{2t\, dt}{t^2 + t} \text{ where } t^2 = x$$

$$= 2 \int \frac{dt}{t+1} = 2 \log (\sqrt{x} + 1) + c$$ **Ans.[B]**

42 $I = \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx$ is equal to-

(A) $\frac{19}{36} x + \frac{35}{36} \log (9e^x - 4e^{-x}) + c$

(B) $-\frac{19}{36} x + \frac{35}{36} \log (9e^x - 4e^{-x}) + c$

(C) $\frac{1}{36} x + \frac{1}{36} \log (9e^x - 4e^{-x}) + c$

(D) None of these

Sol. Suppose $4e^x + 6e^{-x} = A (9e^x - 4e^{-x}) + B (9e^x + 4e^{-x})$

By comparing $4 = 9A + 9B$,

$6 = -4A + 4B$

or $A + B = \frac{4}{9}$, $-A + B = \frac{3}{2}$

After solving $A = -\frac{19}{36}$, $B = \frac{35}{36}$

$$\therefore I = \int\left[-\frac{19}{36}+\frac{35}{36}\left(\frac{9e^{x}+4e^{-x}}{9e^{x}-4e^{-x}}\right)\right]dx$$

$$= -\frac{19}{36}x + \frac{35}{36}\log(9e^{x} - 4e^{-x}) + c$$ **Ans.[B]**

DEFINITE INTEGRATION

43 $\int_0^{\pi/2} |\sin x - \cos x|\, dx$ equals-

(A) $2\sqrt{2}$ (B) $2(\sqrt{2}+1)$

(C) $2(\sqrt{2}-1)$ (D) 0

Sol. $\because |\sin x - \cos x|$

$$= \begin{cases} -(\sin x - \cos x), & 0 < x < \pi/4 \\ (\sin x - \cos x), & \pi/4 < x < \pi/2 \end{cases}$$

$$\therefore I = \int_0^{\pi/4} -(\sin x - \cos x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= [\cos x + \sin x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= 2\sqrt{2} - 2$$ **Ans.[C]**

44 The value of $\lim_{x\to 0} \frac{\int_0^x \cos t^2 dt}{x}$ is-

(A) 0 (B) 1

(C) –1 (D) None of these

Sol. Let $f(x) = \int_0^x \cos t^2 dt$ and $g(x) = x$,

then $f(0) = g(0) = 0$

$$\therefore \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

$$\therefore \text{Given limit} = \lim_{x\to 0} \frac{\cos x^2 .1 - \cos 0.0}{1}$$

$$\left[\text{since} \frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)\,dt = \int_{\phi(x)}^{\psi(x)} \frac{d}{dx}(f(t))\,dt \right.$$

$$\left. = f(\psi(x)\psi'(x) - f\{(\phi(x)\phi'(x)\}\right]$$

$\therefore$ Given limit

$= \cos 0 = 1.$ **Ans.[B]**

45 If $n \in Z$, then

$$\int_0^{\pi} e^{\sin^2 x} \cos^3 (2n+1)x \, dx =$$

(A) -1 (B) 0

(C) 1 (D) π

Sol. Let $f(x) = e^{\sin^2 x} \cos^3 (2n+1)x \, dx$

$\Rightarrow f(\pi - x) = e^{\sin^2 (\pi - x)} \cos^3 (2n+1)(\pi - x) dx$

$= -e^{\sin^2 x} \cos^3 (2n+1)x$

$[\because (2n + 1)$ is odd$]$

$= -f(x)$

So by P – 8, I = 0 **Ans.[B]**

46 $\int_0^1 \frac{6x^2+1}{4x^3+2x+3} dx$ is equal to-

(A) $-\frac{1}{2}\log 3$ (B) $\frac{1}{2}\log 3$

(C) 2 log 3 (D) None of these

Sol. Let $4x^3 + 2x + 3 = t$ $\therefore 2(6x^2 + 1)dx = dt$

Limits –at x = 0; t = 3, at x = 1; t = 9

$\therefore I = \int_3^9 \frac{1}{2}\frac{dt}{t} = \frac{1}{2}[\log t]_3^9$

$= \frac{1}{2}[\log 9 - \log 3] = \frac{1}{2}\log 3$ **Ans.[B]**

47 $\int_0^1 \frac{x}{1+x^4} dx$ is equal to -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{8}$ (D) π

Sol. $I = \frac{1}{2}\int_0^1 \frac{2x}{1+(x^2)^2} dx$

$= \frac{1}{2}[\tan^{-1} x^2]_0^1$

$= \frac{1}{2}[\tan^{-1} 1 - \tan^{-1} 0]$

$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$ **Ans.[C]**

48 $\int_2^4 \frac{\sqrt{x^2-4}}{x} dx$ is equal to

(A) $2(3\sqrt{3} - \pi)$ (B) $2\sqrt{3} - \pi$

(C) $\frac{2}{3}(3\sqrt{3} - \pi)$ (D) π

Sol. Put x = 2 sec t, then

$$I = \int_0^{\pi/3} \frac{2\tan t}{2\sec t}.2\sec t \tan t\, dt$$

$$= 2\int_0^{\pi/3} \tan^2 t\, dt$$

$$= 2\int_0^{\pi/3} (\sec^2 t - 1)\, dt = 2[\tan t - t]_0^{\pi/3}$$

$$= 2[\sqrt{3} - \pi/3] = \frac{2}{3}(3\sqrt{3} - \pi)$$ **Ans. [C]**

49 $\int_0^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$ is equal to

(A) 2 (B) 1

(C) $\pi/4$ (D) $\pi^2/8$

Sol. $\sqrt{x} = t$, $\frac{1}{\sqrt{x}}\, dx = 2dt$

$$\therefore I = 2\int_0^{\pi/2} \sin t\, dt = 2(-\cos t)_0^{\pi/2} = 2(0 + 1) = 2$$

Ans. [A]

50 If $f(x) = \begin{cases} 2x + 1, & 0 < x < 1 \\ x^2 + 2, & 1 \le x < 2 \end{cases}$, then the value of $\int_0^2 f(x)\, dx$ is-

(A) $-\frac{19}{3}$ (B) $\frac{19}{3}$

(C) $\frac{3}{19}$ (D) None of these

Sol. $\int_0^2 f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^2 f(x)\, dx$

$$= \int_0^1 (2x + 1)\, dx + \int_1^2 (x^2 + 2)\, dx$$

$$= \left[x^2 + x\right]_0^1 + \left[\frac{x^3}{3} + 2x\right]_1^2$$

$$= 2 - 0 + \left(\frac{20}{3} - \frac{7}{3}\right) = \frac{19}{3}$$

Ans.[B]

51 $\int_{-4}^{-5} e^{(x+5)^2}\, dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\, dx$ is equal to-

(A) e^5 (B) e^4

(C) $3e^2$ (D) 0

Sol. Putting x = –t –4 in first integral and

$x = \frac{t}{3} + \frac{1}{3}$ in second integral

$I_1 = \int_{-4}^{-5} e^{(x+5)^2} dx = -\int_0^1 e^{(-t+1)^2} dt = -\int_0^1 e^{(t-1)^2} dt$

$I_2 = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$

$= 3\int_0^1 e^{9(t/3-1/3)^2} dt = \int_0^1 e^{(t-1)^2} dt$

$\therefore I = I_1 + I_2 = 0.$ **Ans.[D]**

52 $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ is equal to

(A) $\pi/2$ (B) $\pi/4$

(C) π (D) 2π

Sol. Using prop. P–4, we have

$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$

Adding it to given integral we have

$2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2$

$\therefore I = \pi/4$ **Ans.[B]**

53 If f(x) is an odd function of x, then $\int_{-\pi/2}^{\pi/2} f(\cos x)\,dx$ is equal to

(A) 0 (B) $\int_0^{\pi/2} f(\cos x)\,dx$

(C) $2\int_0^{\pi/2} f(\sin x)\,dx$ (D) $\int_0^{\pi} f(\cos x)\,dx$

Sol. Here f(cos x) will be even function of x,

$I = \int_{-\pi/2}^{\pi/2} f(\cos x)\,dx = 2\int_0^{\pi/2} f(\cos x)\,dx$

$= 2\int_0^{\pi/2} f(\sin x)\,dx$ **Ans.[C]**

54 The value of the integral $\int_{-4}^{4} (ax^3+bx+c)\,dx$ depend on-

(A) b and c (B) a, b and c

(C) only c (D) a and c

Sol. $I = \int_{-4}^{4} (ax^3+bx)\,dx + \int_{-4}^{4} c\,dx$

$= 0 + 2\int_0^4 c\,dx \quad (\text{by } P-5)$

$= 2c\,[x]_0^4 = 8c$

Hence the value of I depends on c.

Ans.[C]

55 If $f(x) = \dfrac{x\cos x}{1+\sin^2 x}$, then $\int_{-\pi}^{\pi} f(x)\,dx$ equals-

(A) $\pi/4$ (B) $\pi/2$

(C) π (D) 0

Sol. Since $f(-x) = \dfrac{-x\cos(-x)}{1+\sin^2(\pi-x)}$

$= \dfrac{-x\cos x}{1+\sin^2 x} = -f(x)$

$\therefore I = \int_{-\pi}^{\pi} f(x)\,dx = 0$ **Ans.[D]**

56 $\int_0^{\pi/2} \sin^2 x\cos^3 x\,dx$ equals-

(A) 1 (B) 2/5

(C) 2/15 (D) 4/15

Sol. Using Walli's formula, we get

$I = \dfrac{1.2}{5.3.1} = \dfrac{2}{15}$ **Ans.[C]**

57 $\int_{\pi/4}^{3\pi/4} \dfrac{\phi}{1+\sin\phi}\,d\phi$ equals-

(A) $\pi(\sqrt{2}-1)$ (B) $\pi(\sqrt{2}+1)$

(C) $\pi(2-\sqrt{2})$ (D) None of these

Sol. $I = \int_{\pi/4}^{3\pi/4} \dfrac{\phi}{1+\sin\phi}\,d\phi$ ...(1)

$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \dfrac{\pi-\phi}{1+\sin(\pi-\phi)}\,d\phi$ (by P – 8)

$= \int_{\pi/4}^{3\pi/4} \dfrac{\pi-\phi}{1+\sin\phi}\,d\phi$ ...(2)

$2I = \int_{\pi/4}^{3\pi/4} \dfrac{\pi}{1+\sin\phi}\,d\phi = \pi\int_{\pi/4}^{3\pi/4} \dfrac{1-\sin\phi}{\cos^2\phi}\,d\phi$

$= \pi\,[\tan\phi - \sec\phi]_{\pi/4}^{3\pi/4} = 2\pi\,(-\sqrt{2}-1)$

$I = \pi(-\sqrt{2}-1)$ **Ans.[A]**

58 $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to-

(A) 2 (B) – 2

(C) 1/2 (D) – 1/2

Sol. By property [P–8]

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x(\pi-x)} = \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x}$$

Adding it with the given integral

$$2I = \int_{\pi/4}^{3\pi/4} \frac{2dx}{1-\cos^2 x} = 2\int_{\pi/4}^{3\pi/4} \text{cosec}^2 x\, dx$$

$$= -2\,[\cot x]_{\pi/4}^{3\pi/4} = 4$$

$$\Rightarrow I = 2$$

Ans.[A]

59 The value of $\int_0^{\pi/2} \sin^3 x\, dx$ is -

(A) 2/3 (B) 3/2 (C) 0 (D) 4π/3

Sol. We have $I = \int_0^{\pi/2} \sin^3 x\, dx = \frac{(3-1)}{3}\ .\ 1$

$= 2/3$.(Since $n = 3$ is odd). **Ans.[A]**

60 $\lim_{n\to\infty} \left[\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + ... + \frac{1}{n}\right]$ is equal to-

(A) $\frac{\pi}{4} + \frac{1}{2}\log 2$ (B) $\frac{\pi}{4} - \frac{1}{2}\log 2$

(C) $\frac{\pi}{4} - 2\log\frac{1}{2}$ (D) None of these

Sol. $T_r = \frac{n+r}{n^2+r^2} = \frac{1}{n}\left[\frac{\left(1+\frac{r}{n}\right)}{1+\left(\frac{r}{n}\right)^2}\right]$

$\therefore$ given limit $= \int_0^1 \frac{1+x}{1+x^2}\, dx$

$= \left[\tan^{-1} x\right]_0^1 + \left[\frac{1}{2}\log(1+x^2)\right]_0^1 = \frac{\pi}{4} + \frac{1}{2}\log 2$ **Ans.[A]**

61 $\int_0^{\infty} \frac{x^3}{(1+x^2)^{9/2}}\, dx$ is equal to-

(A) 2/35 (B) 3/35

(C) 4/35 (D) None of these

Sol. Put $x = \tan t$, then

$I = \int_0^{\pi/2} \frac{\tan^3 t}{\sec^9 t} \sec^2 t\, dt = \int_0^{\pi/2} \sin^3 t \cos^4 t\, dt = \frac{2.3.1}{7.5.3.1} = \frac{2}{35}$ **Ans.[A]**

62 $\int_0^{\infty} \frac{dx}{1+e^x}$ is equal to-

(A) log 2 – 1 (B) log 2

(C) log 4 – 1 (D) –log 2

Sol. $I = \int_0^{\infty} \frac{e^{-x}}{e^{-x}+1} dx = -\left[\log(e^{-x}+1)\right]_0^{\infty}$

$= -[\log 1 - \log 2] = \log 2$ **Ans.[B]**

63 $\int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x \cos x} dx$ is equal to-

(A) 0 (B) 1

(C) π/2 (D) π/4

Sol. Using P–4, given integral becomes

$I = \int_0^{\pi/2} \frac{\cos(\pi/2 - x) - \sin(\pi/2 - x)}{1+\sin(\pi/2-x)\cos(\pi/2-x)} dx = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\cos x \sin x} dx = -I$

$\Rightarrow 2I = 0 \Rightarrow I = 0$ **Ans.[A]**

64 $\int_0^{\infty} \frac{x \ln x}{\left(1+x^2\right)^2}$ dx equals

(A) 0 (B) log 7

(C) 5 log 13 (D) None of these

Sol. Here $\int_0^{\infty} \frac{x\log x}{(1+x^2)^2} dx = \int_0^{1} \frac{x\log x}{(1+x^2)^2} dx + \int_1^{\infty} \frac{x\log x}{(1+x^2)^2} dx$

$I = I_1 + I_2$

Putting $x = \frac{1}{t}$ in second integrand

$dx = -\frac{1}{t^2} dt$

$\therefore I_2 = \int_1^{0} \frac{\frac{1}{t}\log\left(\frac{1}{t}\right)}{\left(1+\frac{1}{t^2}\right)^2} \left(-\frac{1}{t^2}\right) dt = -\int_0^{1} \frac{t\log t}{(1+t^2)^2} dt = -I_1$

$I = I_2 + I_1 = -I_1 + I_1 = 0$ **Ans.[A]**

65 $\int_0^{\pi} x \sin^4 x \, dx$ is equal to-

(A) 3π/16 (B) $3\pi^2/16$

(C) 16π/3 (D) $16\pi^2/3$

Sol. $I = \int_0^{\pi} x \sin^4 x \, dx$ ...(1)

$I = \int_0^{\pi} (\pi - x)\sin^4(\pi - x)\, dx$

$$I = \int_0^{\pi} (\pi - x)\sin^4 x\,dx \qquad ...(2)$$

$$\therefore\ 2I = \pi \int_0^{\pi} \sin^4 x\,dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \sin^4 x\,dx \text{ [from property P-6]}$$

$$\Rightarrow I = \pi.\ \frac{3.1}{4.2}.\frac{\pi}{2} = \frac{3\pi^2}{16}$$ **Ans.[B]**

66 $\int_1^2 \log x\,dx$ equals-

(A) 2 log 2

(B) $\log\left(\frac{2}{e}\right)$

(C) $\log\left(\frac{4}{e}\right)$

(D) None of these

Sol. $I = \int_1^2 1.\log x\,dx$ equals

(Intetgrating by parts by taking 1 as a second function)

$$= \{x.\log x\}_1^2 - \int_1^2 \left(\frac{1}{x}.\ x\right)dx$$

$$= (2 \log 2 - - 1 \log 1) - [x]_1^2$$

$$= (2 \log 2 -- 0) - (2 - 1)$$

$$= \log 4 - \log e = \log\left(\frac{4}{e}\right)$$ **Ans.[C]**

67 $\int_0^{\pi/2} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}}\,dx$ equals-

(A) 2

(B) π

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{2}$

Sol. $I = \int_0^{\pi/2} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}}\,dx$

$$I = \int_0^{\pi/2} \frac{2^{\sin(\pi/2 - x)}}{2^{\sin(\pi/2-x)} + 2^{\cos(\pi/2-x)}}\,dx$$

$$= \int \frac{2^{\cos x}}{2^{\cos x} + 2^{\sin x}}\,dx$$

$$2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$ **Ans.[C]**

68 $\int_0^x f(t) = x + \int_x^1 t\,f(t)\,dt$ then f(1) is equal to-

(A) $\frac{1}{2}$ (B) 0

(C) 1 (D) $-\frac{1}{2}$

Sol. $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$

$\Rightarrow$ f(x) = 1 + (0 – xf(x)) [diff. w.r.t. x]

$\Rightarrow$ f(x) = 1 – xf(x)

$\Rightarrow$ f(1) = 1– 1.f(x)

$\Rightarrow$ $f(1) = \frac{1}{2}$ **Ans.[A]**

69 If f(3 – x) = f(x) then $\int_1^2 x f(x)\,dx$ equals-

(A) $\frac{3}{2}\int_1^2 f(2-x)\,dx$ (B) $\frac{3}{2}\int_1^2 f(x)\,dx$

(C) $\frac{1}{2}\int_1^2 f(x)\,dx$ (D) None of these

Sol. Let x = 3 – y

$I = \int_2^1 (3-y)f(3-y)(-dy)$

$= \int_1^2 (3-x)f(3-x)\,dx$

$= \int_1^2 (3-x)f(x)\,dx$ [$\because$ f(3 – x) = f(x)]

$= 3\int_1^2 f(x)\,dx - I$

$I = \frac{3}{2}\int_1^2 f(x)\,dx$ **Ans.[B]**

70 $\int_0^1 \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ is equal to-

(A) π/2 (B) π /4

(C) 0 (D) 1

Sol. Put $\sin^{-1} x = t$, $\frac{dx}{\sqrt{1-x^2}} = dt$ then

$\therefore I = \int_0^{\pi/2} t\sin t\,dt = [t(-\cos t)]_0^{\pi/2} + [\sin x]_0^{\pi/2} = 1$

Ans.[C]

71 $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}$ is equal to-

(A) 0
(B) 2
(C) 1
(D) None of these

Sol. $I = \int_{-\pi/2}^{0} \frac{\cos x}{1+e^x} dx + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx = -\int_{\pi/2}^{0} \frac{\cos y}{1+e^{-y}} dy + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x}$

(putting x = – y in first integral)

$= \int_{0}^{\pi/2} \frac{e^y \cos y}{1+e^y} dy + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$

$= \int_{0}^{\pi/2} \frac{e^x \cos x}{1+e^x} dx + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$

$= \int_{0}^{\pi/2} \frac{(e^x+1)\cos x}{1+e^x} dx$

$= \int_{0}^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1$ **Ans.[C]**

72 $\int_{-1}^{1} \frac{\sin x - x^2}{3-|x|} dx$ is equal to-

(A) 0
(B) $2\int_{0}^{1} \frac{\sin x}{3-|x|} dx$
(C) $\int_{0}^{1} \frac{-2x^2}{3-|x|} dx$
(D) $2\int_{0}^{1} \frac{\sin x - x^2}{3-|x|} dx$

Sol. $I = \int_{-1}^{1} \frac{\sin x - x^2}{3-|x|} dx$

$= \int_{-1}^{1} \frac{\sin x}{3-|x|} dx - \int_{-1}^{1} \frac{x^2}{3-|x|} dx$

$= 0 - 2\int_{0}^{1} \frac{x^2}{3-|x|} dx$

[$\because \frac{\sin x}{3-|x|}$ is an odd and $\frac{x^2}{3-|x|}$ is an even function]

$= -2\int_{0}^{1} \frac{x^2}{3-|x|} dx$ **Ans.[C]**

73 $\int_{0}^{2a} \frac{f(x)}{f(x)+f(2a-x)} dx$ is equal to-

(A) a
(B) –a
(C) 0
(D) None of these

Sol. Using P–4, given integral becomes

$I = \int_{0}^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)} dx$

Adding it with the given integral, we get

$$2I = \int_0^{2a} 1\,dx = 2a \Rightarrow I = a$$ **Ans.[A]**

74 If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ is equal to-

(A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$

(C) $g(x)\,g(\pi)$ (D) $g(x)/g(\pi)$

Sol. $g(x+\pi) = \int_0^{\pi+x} \cos^4 t\,dt$

$= \int_0^{\pi} \cos^4 t\,dt + \int_{\pi}^{\pi+x} \cos^4 t\,dt$ [by P–3]

$= \int_0^{\pi} \cos^4 t\,dt + I_2$

Now in I_2, put $t = \pi + \theta$, then

$I_2 = \int_0^{x} \cos^4(\pi+\theta)\,d\theta = \int_0^{x} \cos^4 \theta\,d\theta = \int_0^{x} \cos^4 t\,dt$

$\therefore g(x+\pi) = \int_0^{\pi} \cos^4 t\,dt + \int_0^{x} \cos^4 t\,dt = g(x) + g(\pi)$ **Ans.[A]**

75 The value of $\int_0^{100\pi} \sqrt{1-\cos 2x}\,dx$ is

(A) $100\sqrt{2}$ (B) $200\sqrt{2}$

(C) $50\sqrt{2}$ (D) 0

Sol. $I = \sqrt{2}\int_0^{100\pi} |\sin x|\,dx$

$= 100\sqrt{2}\int_0^{\pi} |\sin x|\,dx$

$= 100\sqrt{2}\int_0^{\pi} \sin x\,dx = 100\sqrt{2}\,[-\cos x]_0^{\pi}$

$= 200\sqrt{2}$ **Ans.[B]**

76 $\int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$ is equal to-

(A) $\pi/2$ (B) $\pi/\sqrt{2}$

(C) $-\pi/2$ (D) $-\pi/\sqrt{2}$

Sol. Putting $\tan x = t^2$, then

$\sec^2 x\,dx = 2t\,dt \Rightarrow dx = \frac{2t\,dt}{1+t^4}$

$$\therefore I = \int_0^1 \left(t+\frac{1}{t}\right)\frac{2t\,dt}{1+t^4}$$

$$= 2\int_0^1 \frac{t^2+1}{t^4+1}dt = 2\int_0^1 \frac{1+1/t^2}{t^2+1/t^2}dt = 2\int_0^1 \frac{d(t-1/t)}{(t-1/t)^2+2}$$

$$= \sqrt{2}\left[\tan^{-1}\frac{1}{\sqrt{2}}\left(t-\frac{1}{t}\right)\right]_0^1 = \sqrt{2}\,[\tan^{-1}0 - \tan^{-1}(-\infty)] = \sqrt{2}\,(\pi/2) = \pi/\sqrt{2}$$

Ans.[B]

77 $\int_0^{\pi/2} \frac{dx}{1+2\sin x+\cos x}$ equals-

(A) (1/2) log 3 (B) log 3

(C) (4/3) log 3 (D) None of these

Sol. Here

$$I = \int_0^{\pi/2} \frac{dx}{1+2\frac{2\tan(x/2)}{1+\tan^2(x/2)}+\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}$$

$$= \int_0^{\pi/2} \frac{\sec^2(x/2)}{2\{1+2\tan(x/2)\}}dx$$

Let 1+ 2 tan (x/2) = t, then

$\sec^2(x/2)\,dx = dt$

$$\therefore I = \frac{1}{2}\int_1^3 \frac{dt}{t} = \frac{1}{2}(\log t)_1^3$$

$$= \frac{1}{2}\log 3$$

Ans.[A]

78 $\int_0^{\pi/2} \frac{\sin 2x}{a\cos^2 x+b\sin^2 x}dx$ -

(A) $\frac{1}{b-a}\log\left(\frac{b}{a}\right)$ (B) $\frac{1}{b+a}\log\left(\frac{b}{a}\right)$

(C) $\frac{1}{b-a}\log\left(\frac{a}{b}\right)$ (D) $\frac{1}{b+a}\log\left(\frac{a}{b}\right)$

Sol. $I = \left(\frac{1}{b-a}\right)\int_0^{\pi/2} \frac{(b-a)2\sin x\cos x}{a\cos^2 x+b\sin^2 x}dx$

$$= \frac{1}{b-a}\left[\log(a\cos^2 x+b\sin^2 x)\right]_0^{\pi/2} = \frac{1}{(b-a)}(\log b - \log a)$$

$$= \frac{1}{b-a}\log\left(\frac{b}{a}\right)$$

Ans.[A]

79 $\int_0^{\pi/2}(2\log\sin x - \log\sin 2x)\,dx$ equals-

(A) π log 2 (B) –π log 2

(C) (π/2) log 2 (D) – (π/2) log 2

Sol. $I = \int_0^{\pi/2} (2\log\sin x - \log 2\sin x\cos x)\, dx$

$= \int_0^{\pi/2} (2\log\sin x - \log 2 - \log\sin x - \log\cos x)\, dx$

$= \int_0^{\pi/2} \log\sin x\, dx - \int_0^{\pi/2} \log 2dx - \int_0^{\pi/2} \log\cos x\, dx = -(\pi/2)\log 2.$ **Ans.[D]**

80 $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx$ equals-

(A) $\frac{\pi}{2} + \log 2$ (B) $\frac{\pi}{2} - \log 2$

(C) $\pi - \log 2$ (D) None of these

Sol. $I = \int_0^1 \tan^{-1}\left(\frac{1}{1 - x - x^2}\right) dx$

$= \int_0^1 \tan^{-1}\left(\frac{x + (1-x)}{1 - x(1-x)}\right) dx$

$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]\, dx$

$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$

$= 2\int_0^1 \tan^{-1} x\, dx$ [By prov. IV]

$= 2\left[x\tan^{-1} - \frac{1}{2}\log(1 + x^2)\right]_0^1$

$= 2\frac{\pi}{4} - \log 2 = \frac{\pi}{2} - \log 2$ **Ans.[B]**

EXERCISE 1(B)

More than one options may be correct

1 $\int_0^1 \frac{\sin^{-1} x}{x} dx$ is not equal to-

(*A) $\int_0^{\pi/2} ln\,(\sin\theta)\,d\theta$ (B) $-\int_0^{\pi/2} ln\,(\sin\theta)\,d\theta$ (C) $\int_0^{\pi/2} \theta \cot\theta\, d\theta$ (*D) $ln2 \int_{\pi/2}^{0} d\theta$

Sol. $\int_0^1 \frac{\sin^{-1} x}{x} dx$ Let $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$

$= \int_0^{\pi/2} \frac{\theta}{\sin\theta}.\cos\theta\, d\theta = \int_0^{\pi/2} at\theta\, d\theta$

$= \theta.\ln(\sin\theta)\Big|_0^{\pi/2} - \int_0^{\pi/2} \ln(\sin\theta)\, d\theta$

$= 0 + \frac{\pi}{2}\ln 2$ **Hence (A) & (D)**

2 If $f(x) + f(14 - x) = 4$, and $F(x) = \int_{3-x}^{11+x} f(t)\, dt$ then-

(A) $y = F(x)$ is an expression of degree two.

(*B) $y = F(x)$ represents a straight line.

(*C) $F'(x) = 4$ at $x = 20$

(*D) $F(20) = 96$

Sol. $F(x) = \int_{3-x}^{11+x} f(t)\, dt = \int_{3-x}^{11+x} f(14-t)\, dt$

$2F(x) = \int_{3-x}^{11+x} 4\, dt \Rightarrow$ $F(x) = 2\{8 + 2x\}$ **Hence (B)(C) (D)**

3 $\int \frac{dx}{(ax+b)\sqrt{x}}$ is equal to-

(*A) $-\frac{2}{a\sqrt{x}} + C$ if $b = 0$ and $a \neq 0$

(B) $-\frac{2\sqrt{x}}{b} + C$ if $a = 0$ & $b \neq 0$

(*C) $\frac{2}{\sqrt{ab}}\tan^{-1}\left(\sqrt{\frac{ax}{b}}\right)+C$ if $\frac{a}{b}>0$

(*D) $\frac{1}{\sqrt{-ab}}\ln\left|\frac{\sqrt{x}+\lambda}{\sqrt{x}-\lambda}\right|+C$ where $\lambda^2=-\frac{b}{a}$, if $\frac{b}{a}<0$

Sol. Let $x = t^2 \Rightarrow dx = 2t\,dt$

$$\int\frac{dx}{(ax+b)\sqrt{x}}=2\int\frac{dt}{at^2+b}=\begin{cases}-\frac{2}{t}+C & b=0,\,a\neq 0\\ \frac{2t}{b}+C & a=0,\,b\neq 0\\ \frac{2}{\sqrt{ab}}\tan^{-1}\left(t\sqrt{\frac{a}{b}}\right)+C & \frac{a}{b}>0\\ \frac{1}{\sqrt{-ab}}\ln\left(\frac{t+\sqrt{-\frac{b}{a}}}{t-\sqrt{-\frac{b}{a}}}\right)+C & \frac{b}{a}<0\end{cases}$$

Hence (A), (C), (D)

4 Let $I_n=\int_0^{\pi}(\sin x)^n dx,\ n\in N$

(*A) I_n is a decreasing sequence

(*B) I_n is irrational when n is even

(*C) I_n is rational when n is odd

(*D) $\frac{8}{\pi}(I_2+I_4)=7$

Sol. $I_n=2\int_0^{\pi/2}(\sin x)^n\,dx$

$I_1=2,\ I_2=\frac{\pi}{2},\ I_3=\frac{4}{3},\ I_4=\frac{3\pi}{8}$ **Hence (A) (B) (C) (D)**

5 $\int_0^1\left(\prod_{r=1}^{10}(x+r)\right)\left(\sum_{r=1}^{10}\frac{1}{x+r}\right)dx$ is equal to λ, then

(*A) number of zeros at the end of λ is 3

(B) number of zeros at the end of λ is 4

(C) λ = 11.10!

(*D) λ = 10.10!

Sol. $\int_0^1\left(\prod_{r=1}^{10}(x+r)\right)\left(\sum_{r=1}^{10}(x+r)\right)dx$

$=\prod_{r=1}^{10}(x+r)\Big|_0^1$ = 11! – 10! = 10.10!

Number of zeros at end of λ = 2 + 1 = 3 **Hence (A) (D)**

6 The value of $\int_0^1 \frac{2x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}\,dx$ is :

(A*) $\frac{\pi}{4} + 2\ln 2 - \tan^{-1} 2$ (B) $\frac{\pi}{4} + 2\ln 2 - \tan^{-1}\frac{1}{3}$

(C*) $2\ln 2 - \cot^{-1} 3$ (D*) $-\frac{\pi}{4} + \ln 4 + \cot^{-1} 2$

[Hint: Numerator = $2(x^2 + 2x + 2) - (x + 1)$]

7 $\int \frac{1}{x^2-1}\,\ell n\frac{x-1}{x+1}\,dx$ equals :

(A) $\frac{1}{2}\ln^2\frac{x-1}{x+1} + c$ (B*) $\frac{1}{4}\ln^2\frac{x-1}{x+1} + c$ (C) $\frac{1}{2}\ln^2\frac{x+1}{x-1} + c$ (D*) $\frac{1}{4}\ln^2\frac{x+1}{x-1} + c$

[Hint : put $\ln(x-1) - \ln(x+1) = t$]

8 If $I_n = \int_0^1 \frac{dx}{\left(1+x^2\right)^n}$; $n \in N$, then which of the following statements hold good ?

(A*) $2n\,I_{n+1} = 2^{-n} + (2n-1)\,I_n$ (B*) $I_2 = \frac{\pi}{8} + \frac{1}{4}$

(C) $I_2 = \frac{\pi}{8} - \frac{1}{4}$ (D) $I_3 = \frac{\pi}{16} - \frac{5}{48}$

[Hint: I.B.P. taking 1 as the 2^{nd} and $\frac{1}{\left(1+x^2\right)^n}$ as the 1^{st} function]

9 $\int_0^\infty \frac{x}{(1+x)\left(1+x^2\right)}\,dx$:

(A*) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C*) is same as $\int_0^\infty \frac{dx}{(1+x)\left(1+x^2\right)}$ (D) cannot be evaluated

[Hint : Put $x = 1/t$ and add the two integrals]

10 If $f(x) = \int_0^{\pi/2} \frac{\ell n\left(1 + x\sin^2\theta\right)}{\sin^2\theta}\,d\theta$, $x \ge 0$ then :

(A*) $f(t) = \pi\left(\sqrt{t+1} - 1\right)$ (B*) $f'(t) = \frac{\pi}{2\sqrt{t+1}}$

(C) f (x) cannot be determined (D) none of these.

[Sol. $f'(x) = \frac{dI}{dx} = \int_0^{\pi/2} \frac{\sin^2\theta}{\sin^2\theta(1+x\sin^2\theta)}\,d\theta = \frac{dI}{dx} = \int_0^{\pi/2} \frac{d\theta}{1+x\sin^2\theta}$

Multiply N^r. and D^r. by $\sec^2\theta$ and proceed]

11 If $a, b, c \in R$ and satisfy $3a + 5b + 15c = 0$, the equation $ax^4 + bx^2 + c = 0$ has :

(A*) atleast one root in $(-1, 0)$ (B*) atleast one root in $(0, 1)$

(C*) atleast two roots in $(-1, 1)$ (D) no root in $(-1, 1)$

[Hint : $\int_0^1 f(x)\,dx = \frac{a}{5} + \frac{b}{3} + c = \frac{1}{15}(3a + 5b + 15c) = 0$

$\Rightarrow$ B Since f (x) is even $\Rightarrow$ A $\Rightarrow$ C]

12 Let $u = \int_0^{\infty} \frac{dx}{x^4+7x^2+1}$ & $v = \int_0^{\infty} \frac{x^2\,dx}{x^4+7x^2+1}$ then :

(A) v > u (B*) 6 v = π (C*) 3u + 2v = $5\pi/6$ (D*) u + v = $\pi/3$

[Hint: put x = 1/t in u or v $\Rightarrow$ u = v. Now consider u + v]

13 If $f(x) = \int_1^x \frac{\ell n\, t}{1+t}\,dt$ where x > 0 then the value(s) of x satisfying the equation,

f(x) + f(1/x) = 2 is :

(A) 2 (B) e (C*) e^{-2} (D*) e^2

[Hint: $f(x) = \frac{\ell n^2 x}{2} = 2 \Rightarrow$ C, D]

14 Let $f'(x) = 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}$, if $x \neq 0$; f(0) = 0 and f(1/π) = 0 then

(A*) f(x) is continuous at x = 0 (B) f(x) is non derivable at x = 0

(C*) f ′ (x) is continuous at x = 0 (D*) f ′ (x) is non derivable at x = 0

[Hint: $f(x) = \begin{cases} x^3 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$]

15 If $\int e^u.\sin 2x\,dx$ can be found in terms of known functions of x then u can be :

(*A) x (*B) sinx (*C) cosx (*D) cos 2x

Sol. $\int e^x.\sin 2x\,dx$, $\int e^{\sin x}.\sin 2x\,dx$, $\int e^{\cos x}.\sin 2x\,dx$, $\int e^{\cos 2x}.\sin 2x\,dx$

all can be evaluated **Hence (A) (B) (C) (D)**

16 Let f (x) = tan x – $\tan^3 x + \tan^5 x - \tan^7 x$ + ... ∞, where $x \in \left(0, \frac{\pi}{4}\right)$, then which of the following is / are correct?

(A*) $\int_0^{\pi/6} f(x)\,dx = \frac{1}{8}$ (B) $f'\left(\frac{\pi}{12}\right) = \frac{1}{2}$

(C*) $\lim_{x\to 0^+} \frac{f(x)}{x} = 1$ (D) f (x) is an odd function

[Sol. $f(x) = \frac{\tan x}{1+\tan^2 x} = \frac{1}{2}\sin 2x$

Now verify the alternatives.]

17 Which of the following statement(s) is/are **TRUE?**

(A*) If function y = f(x) is continuous at x = c such that f(c) $\neq$ 0 then f(x) f(c) > 0 $\forall$ x $\in$ (c – h, c + h) where h is sufficiently small positive quantity.

(B) $\lim_{n\to\infty} \frac{1}{n}\, \ell n\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)......\left(1+\frac{n}{n}\right)\right)$ = 1 + 2 ℓn 2.

(C*) Let f be a continuous and non-negative function defined on [a. b].

If $\int_a^b f(x)dx = 0$ then f(x) = 0 $\forall$ x $\in$ [a,b].

(D*) Let f be a continuous function defined on [a, b] such that $\int_a^b f(x)dx = 0$, then there exists atleast one c $\in$ (a, b) for which f(c) = 0.

[Sol.

(A) The expression f(x) f(c) $\forall$ x $\in$ (c – h, c + h) where h $\to 0^+$ is equivalent to $\lim_{x\to 0}$ f(x) f(c) which equals to $(f(c))^2$ because f(x) is continuous.

$\therefore$ f(x) f(c) > 0 $\forall$ x $\in$ (c – h, c + h) where h $\to 0^+$.

(B) We have $I = \lim_{n\to\infty} \frac{1}{n} \ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)......\left(1+\frac{n}{n}\right)\right] = \lim_{n\to\infty} \frac{1}{n} \ln \prod_{k=1}^{n}\left(1+\frac{k}{n}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1+\frac{k}{n}\right) = \int_1^2 \ln x \, dx = \left[x(\ln x - 1)\right]_{x=1}^{x=2} = -1 + 2\ln 2 \approx -0.4.$

(C) Given f(x) $\geq$ 0 $\Rightarrow \int_a^b f(x)dx \geq 0$.

But given $\int_a^b f(x)dx = 0$, so this can be true only when f(x) = 0.

(D) $\int_a^b f(x)\,dx = 0 \Rightarrow$ y = f(x) cuts x axis at least once.

So there exists at least one c $\in$ (a, b) for which f(c) = 0.]

18 If f : R $\to$ R be a continuous function such that $f(x) = \int_1^x 2t f(t)dt$,

then which of the following does not hold(s) good?

(A*) f (π) = e^{π^2} (B*) f (1) = e (C*) f (0) = 1 (D*) f (2) = 2

[Sol. $\because$ f '(x) = 2x f(x) $\Rightarrow$ ℓn f(x) = x^2 + c $\Rightarrow$ f(x) = $e^{x^2} e^c$

f(x) = λe^{x^2}

$\because$ f(1) = 0 $\Rightarrow$ 0 = λe $\Rightarrow$ λ = 0

Hence f(x) = 0, $\forall$x $\in$ R $\Rightarrow$ **A, B, C, D**]

19 Let $f(x) = \int_0^x e^{t-[t]}\,dt$ (x > 0), where [x] denotes greatest integer less than or equal to x, is

(A) continuous and differentiable $\forall$ x $\in$ (0, 3]

(B*) continuous but not differentiable $\forall$ x $\in$ (0, 3]

(C) f (1) = e

(D*) f (2) = 2(e – 1)

[Sol. We have $f(x) = \int_0^x e^{t-[t]}dt = \int_0^x e^{\{t\}}dt$, so

$$f(x) = \begin{cases} \int_0^x e^t\,dt & \text{if } x \in [0,1) \\ \int_0^1 e^t\,dt + \int_1^x e^{t-1}\,dt & \text{if } x \in [1,2) \\ \int_0^1 e^t\,dt + \int_1^2 e^{t-1}\,dt + \int_2^x e^{t-2}\,dt & \text{if } x \in [2,3) \end{cases} \Rightarrow f(x) = \begin{cases} e^x - 1 & \text{if } x \in [0,1) \\ (e-1) + (e^{x-1} - 1) & \text{if } x \in [1,2) \\ 2(e-1) + (e^{x-2} - 1) & \text{if } x \in [2,3) \end{cases}$$

Clearly f(x) is continuous $\forall\, x > 0$ but not differentiable $\forall\, x \in N$ $\Rightarrow$ **(B)**
Also f (2) = 2(e – 1) = 0 = 2(e – 1) $\Rightarrow$ **(D)**]

20 $\int \frac{(\sin x + 2\sin^2 x\cos x) + \cos x(1 + 2\sin 2x) - 2\sin^3 x}{\sin x(1 + \sin 2x)}\,dx$ equals

(A) $-x + \ln|\sin x| + 2\ln|\sin x + \cos x| + c$

(*B) $x + \ln|\sin x| + 2\ln|\sin x + \cos x| + c$

(C) $\ln|\sin x(1 + \sin 2x)| - x + C$

(*D) $\ln|\sin x(1 + \sin 2x)| + x + C$

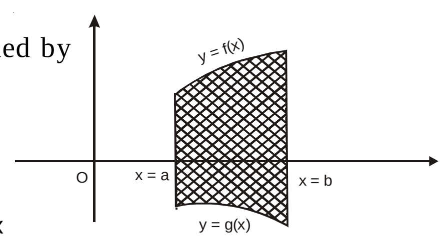
Sol. $\int \frac{(\sin x + 2\sin^2 x\cos x) + \cos x(1 + 2\sin 2x) - 2\sin^3 x}{\sin x(1 + \sin 2x)}\,dx$

$\int \frac{\sin x(1 + \sin 2x) + \cos x(1 + \sin 2x) + 2\sin x(2\cos^2 x - 1)}{\sin x(1 + \sin 2x)}\,dx$

$= x + \ln|\sin x| + \ln|1 + \sin x| + c$

$= x + \ln|\sin x| + 2\ln|\sin x + \cos x| + c$

21 In a given figure, area of shaded region can be obtained by



(*A) $\int_a^b |f(x) - g(x)|\,dx$ (B) $\int_a^b |f(x) + g(x)|\,dx$

(*C) $\int_a^b [|f(x)| + |g(x)|]\,dx$ (D) $\int_a^b [|f(x)| - |g(x)|]\,dx$

Sol. $f(x) > 0$, $g(x) < 0$ for $\forall\, x \in (a, b)$

and $f(x) > g(x)$ Hence (A)

$|f(x)| = f(x)$ and $|g(x)| = -g(x)$ Hence (C)

Hence (A) (C)

22 Let $I_n = \int_0^{\sqrt{3}} \frac{dx}{1+x^n}$ (n = 1, 2, 3 ......) and $\lim_{n\to\infty} I_n = I_0$ (say), then which of the following statement(s) is/are correct? (Given : e = 2.71828)

(A*) $I_1 > I_0$ (B) $I_2 < I_0$ (C*) $I_0 + I_1 + I_2 > 3$ (D*) $I_0 + I_1 > 2$

[Sol. We have $I_1 = \ln(1+\sqrt{3})$

$$I_2 = \frac{\pi}{3}$$

$$I_0 = \lim_{n\to\infty} I_n = \lim_{n\to\infty}\left(\int_0^1 \frac{dx}{1+x^n} + \underbrace{\int_1^{\sqrt{3}} \frac{dx}{1+x^n}}_{\text{zero}}\right) = \int_0^1 dx = 1$$

Hence $I_0 = 1$. Now verify all alternatives.

PASSAGE 1

Let f (x) be a twice differentiable function defined on (– ∞, ∞) such that f (x) = f (2 – x) and $f'\left(\frac{1}{2}\right) = f'\left(\frac{1}{4}\right) = 0$. Then

23 The minimum number of values where f ''(x) vanishes on [0, 2] is

(A) 2 (B) 3 (C*) 4 (D) 5

24 $\int_{-1}^{1} f'(1+x)\, x^2 e^{x^2}\, dx$ is equal to

(A) 1 (B) π (C) 2 (D*) 0

25 $\int_0^1 f(1-t)e^{-\cos\pi t}dt - \int_1^2 f(2-t)e^{\cos\pi t}dt$ is equal to

(A*) $\int_0^2 f'(t)e^{\cos\pi t}dt$ (B) 1 (C) 2 (D) π

[Sol. (1)

∴ $f(x) = f(2-x) \Rightarrow f'(x) = -f'(2-x)$ ....(1)

Putting $x = \frac{1}{2}, \frac{1}{4}$ we get

$$f'\left(\frac{3}{2}\right) = f'\left(\frac{7}{4}\right) = 0$$

Putting x = 1 in (1)

$f'(1) = -f'(1) \Rightarrow f'(1) = 0$

∴ $f'(x) = 0$ will have atleast five real roots in [0, 2]

∴ $f''(x) = 0$ will have at least four real roots in [0, 2]

(2)

Replacing x by 1 + x in (1), we get

$f'(1+x) = -f'(1-x)$

Let $\quad I = \int_{-1}^{1} f'(1+x)\, x^2 e^{x^2}\, dx$ ....(2)

$I = \int_{-1}^{1} f'(1-x) \cdot x^2 e^{x^2}\, dx$

$I = -\int_{-1}^{1} f'(1+x) \cdot x^2 e^{x^2}\, dx$ ....(3) $\quad (\because f'(1+x) = -f'(1-x))$

from (2) + (3), we get $2I = 0 \Rightarrow I = 0$

(3)

Let $\quad I = \int_0^1 f(1-t) e^{-\cos \pi t}\, dt - \int_1^2 f(2-t) e^{\cos \pi t}\, dt$

$= \int_0^1 f(1-(1-t)) e^{-\cos \pi (1-t)}\, dt - \int_1^2 f(2-t) e^{\cos \pi t} dt$ (in Ist)

$= \int_0^1 f(t) e^{\cos \pi t} dt - \int_1^2 f(t) e^{\cos \pi t} dt$ $\quad (\therefore \; f(2-t) = f(t))$

$\because \quad \int_0^2 f(t) e^{\cos \pi t} dt = 2\int_0^1 f(t) e^{\cos \pi t} dt$ $\quad (\therefore \; f(2-t) e^{\cos \pi (2-t)} = f(t) e^{\cos \pi t})$

$\Rightarrow \quad \int_1^2 f(t) e^{\cos \pi t} dt = \int_0^1 f(t) e^{\cos \pi t} dt$

$\therefore \quad I = 0$

$\int_0^2 f'(t) e^{\cos \pi t} = 0$ $\quad \{\therefore \; f'(2-t) = -f'(t)\}$]

PASSAGE 2

Let $g: R \to R$ be a differentiable function which satisfies $g(x) = 1 + \int_0^x g(t)\, dt$ and $g'(0) = 1$

26 The value of $g(ln\, 10) + g'(ln\, 10) + g''(ln\, 10)$ is equal to

(A) 0 (B) $\frac{1}{10}$ (C*) 30 (D) $\frac{1}{30}$

27 The value of definite integral $\int_{-3}^{-1} \left(\sum_{r=1}^{\infty} g(rx) \right) dx$ is equal to

(A) $ln\,(1 + e + e^{-1})$ (B*) $ln\,(1 + e^{-1} + e^{-2})$

(C) $ln\,(1 + e + e^{2})$ (D) $(1 + e^{-1} + e^{2})$

28 Number of solution of the equation $f(-x) = f(x)$ is equal to

(A) 0 (B*) 1 (C) 2 (D) 3

[Sol. We have $g(x) = 1 + \int_0^x g(t)\,dt$ ....(1)

Now, on differentiating both the sides of equation (1) with respect to x, we get

$g'(x) = g(x)$ ....(2)

But $g(x) = 0$ (Not possible as $g(0) = 1$)

So, $\int \frac{g'(x)}{g(x)}dx = \int 1\,dx \Rightarrow \ln(g(x)) = x + A$

$\therefore$ $A = 0$ (As $g(0) = 1$)

Hence $g(x) = e^x$

(i) Hence $g(\ln 10) + g'(\ln 10) + g''(\ln 10) = 10 + 10 + 10 = 30$

(ii) We have $f(x) + f(2x) + \ldots\ldots \infty = e^x + e^{2x} + e^{3x} \ldots \infty = \frac{e^x}{1-e^x}$

If $x < 0$ then $e^x < 1$.

$$\therefore \int_{-3}^{-1}\left(\sum_{r=1}^{\infty} g(rx)\right)dx = \int_{-3}^{-1}\frac{e^x dx}{1-e^x}$$

$$= \left[-\left(\ln\left(1-e^x\right)\right)\right]_{-3}^{-1}$$

$$= \ln\left(1+\frac{1}{e}+\frac{1}{e^2}\right)$$

(iii) As $f(-x) = f(x)$ gives $e^{-x} = e^x \Rightarrow e^{2x} = 1$

$\therefore$ $x = 0$

Hence number of solution of given equation is one.

PASSAGE 3

Consider the function defined on $[0, 1] \to R$

$$f(x) = \frac{\sin x - x\cos x}{x^2} \text{ if } x \neq 0 \text{ and } f(0) = 0$$

29 $\int_0^1 f(x)\,dx$ equals

(A*) $1 - \sin(1)$ (B) $\sin(1) - 1$ (C) $\sin(1)$ (D) $-\sin(1)$

[Sol. $\int_0^1 \frac{\sin x}{x^2}dx - \int_0^1 \frac{\cos x}{x}dx = \sin x\left(-\frac{1}{x}\right)\Big|_0^1 + \int_0^1 \cos x\frac{1}{x}dx - \int_0^1 \frac{\cos x}{x}dx$

$= -\left[\frac{\sin x}{x}\right]_0^1 = (1) - \sin(1)$ Ans.]

30 $\lim\limits_{t\to 0} \frac{1}{t^2}\int_0^t f(x)\,dx$ equals

(A) 1/3 (B*) 1/6 (C) 1/12 (D) 1/24

[Sol. $\lim\limits_{t\to 0} \frac{\int_0^t f(x)\,dx}{t^2} = \lim\limits_{t\to 0} \frac{\int_0^t \frac{\sin x - x\cos x}{x^2}dx}{t^2}$

using L'Hospital's rule

$$l = \lim_{t\to 0} \frac{\sin t - t\cos t}{t^2 \cdot 2t}$$

$$= \lim_{t\to 0} \frac{\cos t(\tan t - t)}{2t^3}$$

$$= \frac{1}{2}\lim_{t\to 0} \frac{\sec^2 t - 1}{3t^2} = \frac{1}{6}$$

PASSAGE 4

Definte integral of any discontinuous or non-differentiable function is normally solved by the property $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $c \in (a,b)$ is the point of discontinuity or non-differentiability.

31 The value of $A = \int_1^\infty [\operatorname{cosec}^{-1} x]dx$, {where [.] denotes greatest integer function}, is equal to

(A*) $\operatorname{cosec}1 - 1$ (B) 1 (C) $1 - \sin 1$ (D) none of these

Sol $A = \int_1^\infty [\operatorname{cosec}^{-1} x]dx$,

$$= \int_1^{\operatorname{cosec}1} 1\,dx + \int_{\operatorname{cosec}1}^{\infty} 0\,dx$$

32 The value of $B = \int_1^{100} [\sec^{-1} x]dx$,{where [.] denotes greatest integer function}, is equal to

(A) sec 1 (B*) $100 - \sec 1$ (C) $99 - \sec 1$ (D) none of these

Sol $\int_1^{100} [\sec^{-1} x]dx = \int_1^{\sec 1} 0\,dx + \int_{\sec 1}^{100} 1dx = 100 - \sec 1$

33 The value of integral $\int_A^B [\tan^{-1} x]dx$, {where [.] denotes greatest integer function},is equal to

(A) tan 1 (B*) $100 - \tan 1 - \sec 1$

(C) $99 - \sec 1$ (D) none of these

Sol $\int_{\operatorname{cosec}1-1}^{100-\sec 1} [\tan^{-1} x]dx = 100 - \tan 1 - \sec 1$

Assertion reasoning

34 **Statement-1:** Let $I_n = \int_0^1 (1-x^5)^n\,dx$. Then $\frac{I_{10}}{I_{11}} = \frac{55}{56}$.

Statement-2: If u (x) and v (x) are differentiable function, then $\int u\,dv = uv - \int v\,du + C$, where C is constant of integration .

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D*) Statement-1 is false, statement-2 is true.

[Sol. $I_{11} = \int_0^1 \underbrace{(1-x^5)^{11}}_{I} \cdot \underset{II}{1}\,dx = (1-x^5)^{11}\cdot x\Big]_0^1 + 11\int_0^1 (1-x^5)^{10} 5x^4 \cdot x\,dx$

$I_{11} = 0 - 55\int_0^1 (1-x^5)^{10}(1-x^5-1)\,dx = -55\int_0^1 (1-x^5)^{11}\,dx + 55I_{10}$

$56I_{11} = 55I_{10} \Rightarrow \frac{I_{10}}{I_{11}} = \frac{56}{55}$]

35 **Statement-1:** If $f(x) = \int_1^x \frac{\ln t}{1+t+t^2}\,dt\,(x>0)$ then $f(x) = -f\left(\frac{1}{x}\right)$

Statement-2: If $f(x) = \int_1^x \frac{\ln t\,dt}{t+1}$ then $f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2}(\ln x)^2$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D*) Statement-1 is false, statement-2 is true.

[Hint $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1+t+t^2}\,dt$; putting $t = \frac{1}{Z}$; $f\left(\frac{1}{x}\right) = f(x)$

36 **Statement 1:** If $x>0, x \ne 1$ then $\int \left(\log_x e - (\log_x e)^2\right)dx = x\log_x e + C$

Statement 2: $\int e^x\left(f(x) + f'(x)\right)dx = e^x f(x) + C$ and $e^t = x$ iff $t = \ell n x$

(A*) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Sol $\int \left(\log_x e - (\log_x e)^2\right)dx$

$= \int \left(\frac{1}{\ell nx} - \frac{1}{(\ell nx)^2}\right)dx = \int \left(\frac{1}{t} - \frac{1}{t^2}\right)e^t\,dt \qquad \{\text{Where } t = \ell n\,x\}$

$= \frac{e^t}{t} + C = \frac{x}{\ell nx} + C = x\log_x e + C$

37 **Statement 1:** $\int 2^{\tan^{-1}x} d\left(\cot^{-1}x\right) = \frac{2^{\tan^{-1}x}}{\ell n 2} + c$ where c is the constant of integration.

Statement 2 : $\frac{d}{dx}\left(a^x + c\right) = a^x \ \ell n\ a$ where c is any constant.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D*) Statement-1 is false, statement-2 is true.

Sol Since $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$,

$\therefore \quad d\left(\cot^{-1}x\right) = -d\left(\tan^{-1}x\right)$

Thus $\int 2^{\tan^{-1}x} d\left(\cot^{-1}x\right) = -\int 2^{\tan^{-1}x} d\left(\tan^{-1}x\right) = -\frac{2^{\tan^{-1}x}}{\ell n\ 2} + c$.

Statement -1 is False
Statement -2 is True.

Match the column

38

	Column-I	**Column-II**	
(A)	Let $f(t) = \sqrt{1-\sin t}$, then $\int_0^{2\pi} f(t)\,dt - \int_0^{\pi} f(t)\,dt$, is equal to	(P)	2
(B)	For $x \neq 2$, if $\int_{4-x}^{x} e^{x(4-x)}dx = 2$, then $\int_{4-x}^{x} x\, e^{x(4-x)}dx$ is equal to	(Q)	4
(C)	Let f be a differentiable function on R satisfying $f(x) = x^2 + \int_1^x t\,f(t)\,dt$. If $f(0) = -1$ then the value of $\frac{f'(2)}{e^2}$ is equal to	(R)	6
		(S)	8

[Ans. (A) Q ; (B) Q; (C) P]

[Sol.

(A) $I = \int_0^{\pi} f(t)\,dt + \int_{\pi}^{2\pi} f(t)\,dt - \int_0^{\pi} f(t)\,dt = \int_{\pi}^{2\pi} \sqrt{1-\sin t}\ dt$

Put $t = \pi + y$, we get

$I = \int_0^{\pi} \sqrt{1-\sin(\pi+y)}\ dy = \int_0^{\pi} \sqrt{1+\sin y}\ dy = \int_0^{\pi} \left|\cos\frac{y}{2} + \sin\frac{y}{2}\right| dy$

Put $\frac{y}{2} = \theta \Rightarrow dy = 2\ d\theta = 2\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)\,d\theta = 4$ **Ans.**]

(B) Let $I = \int_{4-x}^{x} x\, e^{x(4-x)}\ dx$ ....(1)

Also, $I = \int_{4-x}^{x} (4-x)\, e^{x(4-x)} dx$ ....(2)

Adding (1) and (2), we get , $2I = \int_{4-x}^{x} 4\, e^{x(4-x)} dx \Rightarrow 2I = 4 \times 2$, so $I = 4$ **Ans.**]

(C) Differentiate given relation w.r.t 'x' to get $f'(x) = 2x + x f(x) \Rightarrow f'(x) = (2 + f(x))x$

Let $y = f(x)$ then $\frac{dy}{dx} = 2x + xy$ or $\frac{dy}{2+y} = x\, dx$

$\Rightarrow ln(y + 2) = \frac{x^2}{2} + C \Rightarrow y + 2 = e^{\frac{x^2}{2}} \Rightarrow y = \left(e^{\frac{x^2}{2}} - 2\right)$

(As C = 0 beacause f (0) = – 1)

So, $y'(x) = x e^{\frac{x^2}{2}}$

Hence $\frac{y'(2)}{e^2} = 2$

39 Let $I_1 = \int \tan x \tan(ax+b) dx$ and $I_2 = \int \cot x \cot(ax+b) dx$

Column-I	**Column-II**
(A) value of I_1 for a = 1 is	(P) $x - \cot b\, \ell n \frac{\cos(x-b)}{\cos x} + C$
(B) value of I_2 for a = 1 is	(Q) $\cot b\, \ell n \frac{\sin x}{\sin(x+b)} - x + C$
(C) value of I_1 for a = – 1 is	(R) $\cot b\, \ell n \left(\frac{\cos x}{\cos(x+b)}\right) - x + C$
(D) value of I_2 for a = – 1 is	(S) $x + \cot b\, \ell n \left(\frac{\sin x}{\sin(b-x)}\right) + C$

Sol. $I_1 = \int \tan x \tan(ax+b) dx$

(A) for a = 1,

$I_1 = \int \tan x \tan(x+b) dx$

tan b = tan [(x+b) – (x)]

$= \frac{\tan(x+b) - \tan x}{1 + \tan(x+b) \tan x}$

or $\tan(x + b)\tan x = \dfrac{\tan(x+b) - \tan x - \tan b}{\tan b}$

or $I_1 = \dfrac{1}{\tan b}\int (\tan(x+b) - \tan x - \tan b)dx$

$= \dfrac{1}{\tan b}[-\log\cos(x+b) + \log\cos x - x\tan b] + c$

or $\boxed{I_1 = \cot b\,\ell n\left(\dfrac{\cos x}{\cos(x+b)}\right) - x + c}$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(B) $I_2 = \int \cot x \cot(ax+b)dx$

for a = 1

$\cot b = \cot((x + b) - x)$

$\cot b = \dfrac{\cot(x+b)\cot x + 1}{\cot x - \cot(x+b)}$

or $\cot(x + b)\cot x = \cot b \cot x - \cot b \cot(x + b) - 1$

or $I_2 = \int (\cot b \cot x - \cot b \cot(x+b) - 1)\,dx$

$= \cot b \int \cot x dx - \cot b \int \cot(x+b)dx - \int 1.dx$

$= \cot b \log(\sin x) - \cot b \log(\sin(x + b)) - x$

$= \cot b \log \dfrac{\sin x}{\sin(x+b)} - x$

(C) for a = – 1

$I_1 = \int \tan x \tan(b-x)dx$

$\tan b = \tan(x + (b - x))$

$= \dfrac{\tan x + \tan(b-x)}{1 - \tan x \tan(b-x)}$

$\tan x \tan(b - x) = \dfrac{\tan b - \tan x - \tan(b-x)}{\tan b}$

or $I_1 = \dfrac{1}{\tan b}\int (\tan b - \tan x - \tan(b-x))dx$

$= \dfrac{1}{\tan b}[x\tan b + \log\cos x - \log\cos(b-x)] + c$

$= x + \cot b \log \dfrac{\cos x}{\cos(b-x)} + c$

(D) $I_2 = \int \cot x \cot(ax+b)dx$

for a = – 1

cot b = cot (x + (b – x))

$$= \frac{\cot x \cot(b-x) - 1}{\cot x + \cot(b-x)}$$

or cot x cot (b – x) = cot b (cot x + cot (b – x)) + 1

or $I_2 = \int [(\cot b(\cot x + \cot(b-x)) + 1]dx$

$$= \cot b\left[\log \frac{\sin x}{\sin(b-x)}\right] + x$$

40	Column - I	Column - II
(A)	If $I = \int \frac{\sin x - \cos x}{\lvert \sin x - \cos x \rvert} dx$, where $\frac{\pi}{4} < x < \frac{3\pi}{8}$. then I equal to	(P) sin x
(B)	If $\int \frac{x^2}{(x^3+1)(x^3+2)} dx = \frac{1}{3} f\left(\frac{x^3+1}{x^3+2}\right) + C$, then f(x) is equal to	(Q) $x + c$
(C)	If $\int \sin^{-1} x . \cos^{-1} x \, dx = f^{-1}(x)\left[\frac{\pi}{2}x - xf^{-1}(x) - 2\sqrt{1-x^2}\right] + 2x + C$, then f(x) is equal to	(R) $\ell n\lvert x \rvert$
(D)	If $\int \frac{dx}{xf(x)} = f(f(x)) + C$, then f(x) is equal to	(S) $\sin^{-1} x$

Sol. $A \to Q, B \to R, C \to P, D \to R$

(A) If $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then $\sin x > \cos x$

$\therefore \quad \int \frac{\sin x - \cos x}{\lvert \sin x - \cos x \rvert} dx = \int 1.dx = x + c$

(B) $\int \frac{x^2 \, dx}{(x^3+1)(x^3+2)} = \frac{1}{3} \int 3x^2\left(\frac{1}{x^3+1} - \frac{1}{x^3+2}\right)dx = \frac{1}{3}\ell n\left\lvert \frac{x^3+1}{x^3+2} \right\rvert + c$

$\therefore \quad f(x) = \ell n\lvert x \rvert$

(C) $\int \sin^{-1} x \cos^{-1} x \, dx = \int \left[\frac{\pi}{2}\sin^{-1} x - (\sin^{-1} x)^2\right] dx$

$\Rightarrow \quad \frac{\pi}{2}\left(x \sin^{-1} x + \sqrt{1-x^2}\right) - \left(x(\sin^{-1} x)^2 + \sin^{-1} x\sqrt{1-x^2} - x\right) + c$ By parts

$\Rightarrow \quad \sin^{-1} x\left[\frac{\pi}{2}x - x\sin^{-1} x - 2\sqrt{1-x^2}\right] + \frac{\pi}{2}\sqrt{1-x^2} + 2x + c$

$\therefore \quad f^{-1}(x) = \sin^{-1} x, f(x) = \sin x$

(D) $\int \frac{dx}{x\,\ell n|x|} = \ell n|\ell n|x|| + c$

$\therefore \quad f(x) = \ell n|x|$.

EXERCISE 1(C)

Subjective type

1 Consider the polynomial $f(x) = ax^2 + bx + c$. If $f(0) = 0$, $f(2) = 2$,

then the minimum value of $\int_0^2 |f'(x)|dx$ is [Ans. 2]

[Sol. $\int_0^2 |f'(x)|dx \ge \left|\int_0^2 f'(x)dx\right|$; $\int_0^2 |f'(x)|dx \ge |f(2)| = 2$

2 Let $f(x)$ be a continuous function on [0, 4] satisfying $f(x)\ f(4-x) = 1$.

The value of the definite integral $\int_0^4 \frac{1}{1+f(x)}\,dx$ is [Ans. 2]

[Sol. Let $I = \int_0^4 \frac{1}{1+f(x)}\,dx$ .... (1)

Now on applying king property in (1), we get

$I = \int_0^4 \frac{1}{1+f(4-x)}\,dx$, put $f(4-x) = \frac{1}{f(x)}$ $\Rightarrow$ $I = \int_0^4 \frac{f(x)}{f(x)+1}\,dx$ ....(2)

Now (1) + (2) $\Rightarrow$ $2I = \int_0^4 dx$ $\Rightarrow$ $I = 2$

3 Let $T = \int_0^{ln 2} \frac{2e^{3x}+e^{2x}-1}{e^{3x}+e^{2x}-e^x+1}\,dx$, then $e^T = \frac{p}{q}$ where p and q are coprime to each other, then the value of p + q is [Ans. 15]

[Sol. We have $T = \int_0^{ln 2} \frac{(3e^{3x}+2e^{2x}-e^x)-(e^{3x}+e^{2x}-e^x+1)}{e^{3x}+e^{2x}-e^x+1}\,dx = \left[ln\left(e^{3x}+e^{2x}-e^x+1\right)-x\right]_0^{ln 2}$

$= \left(ln(8+4-2+1)-ln 2\right)-\left(ln 2-0\right) = ln\,\frac{11}{2} - ln\,2 = ln\,\frac{11}{4}$ $\Rightarrow$ $e^T = e^{ln\frac{11}{4}} = \frac{11}{4}$

4 If $\int_0^{g(x)} f(t)\,dt = x^2 + \cos \pi x + 1 \quad \forall\, x \ge 1$, where g(x) is inverse of f(x). If f(3) = 4,

then $f'(3) = \frac{p}{q}$ where p and q are coprime to each other, then the value of p + q is

[Ans. 3]

Sol. $\int_1^{g(x)} f(t)\,dt = x^2 + \cos(\pi x) + 1$

$f(g(x)).\ g'(x) = 2x - \pi \sin \pi x$

since f(x) and g(x) are inverse of each other f(g(x)) = x

$x\ g'(x) = 2x - \pi \sin \pi x$

substituting x = 4

$4g'(4) = 8 \Rightarrow g'(4) = 2$

Hence $f'(3) = \frac{1}{2}$

5 Let $f : (0, \infty) \to R$ be a continuous function such that $F(x) = \int_0^x t\ f(t)dt$.

If $F(x^2) = x^4 + x^5$, then the biggest prime factor of the value of $\sum_{r=1}^{12} f(r^2)$ is [Ans. 73]

[Sol. We have $F(x^2) = \int_0^{x^2} t\ f(t)\,dt = x^4 + x^5$ ....(1)

$\therefore$ On differentiating both the sides w.r.t. x, we get
$2x\,(x^2)\,f(x^2) = 4x^3 + 5x^4$

$\Rightarrow$ $f(x^2) = 2 + \frac{5}{2}x$ ....(2)

$\therefore$ $\sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12}\left(2 + \frac{5}{2}r\right) = 24 + \left(\frac{5}{2}\right)\frac{(12)(13)}{2} = 24 + (15)(13) = 24 + 195 = 216$

Hence $\sum_{r=1}^{12} f(r^2) = 219$

6 $I = \int \frac{x-1}{(x-3)(x-2)}dx \equiv A\ \ell n(x-3) + B\ \ell n(x-2) + c$, then find the value of A + B.

[Ans. 1]

Sol. $I = \int \frac{x-1}{(x-3)(x-2)}dx = \int \left[\frac{2}{x-3} - \frac{1}{x-2}\right]dx$

$\Rightarrow$ $I = 2\ \ell n(x-3) - \ell n(x-2) + c$

so $A = 2, B = -1$

$\therefore$ $A + B = 1$

7 Let f(x), g(x) and h(x) are continuous function in [0, a] such that f(a – x) = f(x) , g(a – x) + g(x) = 0 and h(x) + h(a – x) = 3, then the value of $\left|\dfrac{\int_0^a f(x)\cdot g(x)\cdot h(x)dx}{\int_0^a f(x)\cdot g(x)dx}\right|$ is

[Ans. 3]

Sol. $I = \int_0^a f(x).g(x).h(x)\,dx$

$= \int_0^a f(a-x).g(a-x).h(a-x)\,dx$

$= \int_0^a f(x).(-g(x)).(3-2h(x))dx$

$= -3\int_0^a f(x).g(x)\,dx + 2\int_0^a f(x).g(x).h(x)\,dx$

$= -3\int_0^a f(x).g(x)\,dx + 2$

$I = 3\int_0^a f(x).g(x)\,dx$

$= 3\int_0^a f(a-x).g(a-x)\,dx$

8 If f(x) + f(x + 4) = f(x + 2) $\forall x \in R$ and $\int_3^{15} f(x)\,dx = 10$ then find the value of $\int_{10}^{70} f(x)\,dx$ [Ans. 50]

Sol. f(x) + f(x + 4) = f(x + 2) ....(i)

replace x by x + 2

f(x + 2) + f(x + 6) = f(x + 4) ....(ii)

(i) + (ii) $\Rightarrow$ f(x) + f(x + 6) = 0 ....(iii)

replace x by x + 6

f(x + 6) + f(x + 12) = 0 ....(iv)

(iii) – (iv) $\Rightarrow$ f(x) – f(x + 12) = 0

hence f(x) is periodic with period 12

$\int_3^{15} f(x)\,dx = \int_3^{3+12} f(x)\,dx \Rightarrow \int_0^{12} f(x)\,dx = 10$

Also $\int_{10}^{70} f(x)\,dx = \int_{10}^{10+60} f(x)\,dx = 5\int_0^{12} f(x)\,dx = 50$

9 If $\int_0^1 \frac{dx}{2e^x - 1}$ = p ℓn(qe – 1) – 1 , then the value of p + q is [Ans. 3]

Sol. $\int_0^1 \frac{dx}{2e^x-1} = \int_0^1 \left(\frac{2e^x - 2e^x + 1}{2e^x - 1}\right) dx = \int_0^1 \left(\frac{2e^x}{2e^x-1} - 1\right) dx$

$= [\ell n(2e^x - 1) - x]_0^1 = \ell n(2e - 1) - 1$

$\Rightarrow$ p = 1 ; q = 2

10 If $\int_0^{\pi} e^{r\cos x}.\cos(x + r\sin x)\,dx = S$, then the value of S is [Ans. 0]

Sol. cos(x + rsinx) = cos x . cos(r sinx) – sinx. sin(rsinx)

$\Rightarrow \int_0^{\pi} e^{r\cos x}.\cos(x + r\sin x)\,dx$

$= \int_0^{\pi} e^{r\cos x}.\{\cos x.\cos(r\sin x) - \sin x.\sin(r\sin x)\}\,dx$

$= \int_0^{\pi} e^{r\cos x}.\cos(r\sin x).\cos x dx + \frac{1}{r}\int_0^{\pi} e^{r\cos x}(-r\sin x).\sin(r\sin x)\,dx$

$= \int_0^{\pi} e^{r\cos x}.\cos(r\sin x).\cos x dx + \frac{1}{r}\left[\left|e^{r\cos x}.\sin(r\sin x)\right|_0^{\pi} - \int_0^{\pi} e^{r\cos x}\cos(r\sin x) r\cos x dx\right]$

$= \frac{1}{r}\left|e^{r\cos x}.\sin(r\sin x)\right|_0^{\pi} = \frac{1}{r}\left[e^{-r}.(0) - e^{r}.(0)\right] = 0$

11 If the two lines AB : $\left(\int_0^{2t}\left(\frac{\sin x}{x} + 1\right) dx\right)$x + y = 3t and AC : 2t x + y = 0 intersect at a point A, then x-coordinate of point A as t $\to$ 0, is equal to $\frac{p}{q}$ (p and q are in their lowest form).

The value of (p + q) is [Ans. 5]

[Sol. $x_A = \lim_{t\to 0} \frac{3t}{\int_0^{2t}\left(\frac{\sin x}{x}+1\right)dx - \int_0^{2t} 1.dx} = 3\ ;\ \lim_{t\to 0}\frac{t}{\int_0^{2t}\frac{\sin x}{x}dx} = \frac{3}{2\frac{\sin 2t}{2t}} = \frac{3}{2}$

12 If J = $\int_0^{10} \text{sgn}(\sin \pi x)\,dx$, where sgn x denotes signum function of x, then the value of 10J is

[Ans. 0]

Sol. J = $\int_0^{10} \text{sgn}\ (\sin \pi x)\,dx$

$$= 5\int_0^2 \text{sgn}\,(\sin \pi x)\,dx \qquad \text{(As sgn (sin } \pi x\text{) is periodic with fundamental period 2.)}$$

$$= 5\int_0^1 1\,dx + 5\int_1^2 -1\,dx = 5 - 5 = 0$$

13 The value of $\displaystyle\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{x^4 + x^2 + 2}{(x^2+1)^2}\,dx$ is [Ans. 2]

[Sol. Let $x = \tan\theta$

$$= \int_{\pi/8}^{3\pi/8} \frac{\sec^4\theta - \sec^2\theta + 2}{\sec^4\theta}\sec^2\theta d\theta = \int_{\pi/8}^{3\pi/8}\left(\sec^2\theta - 1 + 2\cos^2\theta\right)d\theta = \int_{\pi/8}^{3\pi/8}\left(\sec^2\theta - 1 + 1 + \cos 2\theta\right)d\theta$$

$$= \int_{\pi/8}^{3\pi/8} \sec^2\theta + \cos 2\theta\, d\theta = \left[\tan\theta + \frac{\sin 2\theta}{2}\right]_{\pi/8}^{3\pi/8} = 2$$

14 Given y (0) = 2000 and $\frac{dy}{dx}$ = 32000 – 20y^2, then find the value of $\lim\limits_{x\to\infty} \frac{y(x)}{10}$. [Ans. 4]

[Sol. We have $\frac{dy}{dx} = 20(1600 - y^2)$

$$\Rightarrow \int \frac{dy}{(40)^2 - y^2} = 20\int dx$$

$$\Rightarrow \frac{1}{80} ln \frac{40+y}{40-y} = 20x + C' \quad \text{or} \quad ln \frac{40+y}{40-y} = 1600x + C$$

$$\Rightarrow \frac{40+y}{40-y} = \frac{ke^{1600x}}{1}, \quad \text{where } k = e^C \text{ (let)}$$

$$\Rightarrow \frac{2y}{80} = \frac{ke^{1600x} - 1}{ke^{1600x} + 1} \quad \text{(using componendo \& dividendo)}$$

$$\therefore \lim_{x\to\infty} y = 40 \lim_{x\to\infty}\left[\frac{k - e^{-1600x}}{k - e^{-1600x}}\right] = 40$$

15 A continuous real function f satisfies f (2x) = 3 f (x) $\forall\, x \in R$

If $\int_0^1 f(x)\,dx = 1$, then the value of definite integral $\int_1^2 f(x)\,dx$ is [Ans. 5]

[Sol. We have f (2x) = 3 f (x) ....(1)

and $\int_0^1 f(x)\,dx = 1$ ....(2)

From (1) and (2), $\frac{1}{3}\int_0^1 f(2x)\,dx = 1$

Put 2x = t, $\frac{1}{6}\int_0^2 f(t)\,dt = 1 \Rightarrow \int_0^2 f(t)\,dt = 6 \Rightarrow \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 6$

Hence $\int_1^2 f(t)\,dt = 6 - \int_0^1 f(t)\,dt = 6 - 1 = 5$

16 Consider a polynomial P(x) of the least degree that has a maximum equal to 6 at x = 1, and a minimum equal to 2 at x = 3. The value of P(2) + P'(0) is [Ans. 13]

[Sol. The polynomial is an everywhere differentiable function. Therefore, the points of extremum can only be roots of the derivative. Furthermore, the derivative of a polynomial is a polynomial. The polynomial of the least degree with roots $x_1 = 1$ and $x_2 = 3$ has the from $a(x - 1)(x - 3)$.

Hence $P'(x) = a(x - 1)(x - 3) = a(x^2 - 4x + 3)$ since at the point x = 1, there must be P(1) = 6, we have

$$P(x) = \int_1^x P'(x)dx + 6 = a\int_1^x (x^2 - 4x + 3)dx + 6 = a\left(\frac{x^3}{3} - 2x^2 + 3x - \frac{4}{3}\right) + 6$$

The coefficient 'a' is determined from the condition P(3) = 2, whence a = 3.

Hence $P(x) = x^3 - 6x^2 + 9x + 2$

Now P(2) = 8 – 24 + 18 + 2 = 28 – 24 = 4

Also $P'(x) = 3(x^2 - 4x + 3) \Rightarrow P'(0) = 9$

$\therefore$ P(2) + P'(0) = 4 + 9 = 13

17 If $f(x) = x + \int_0^1 t(x+t)f(t)\,dt$, then the value of the definite integral $\int_0^1 f(x)\,dx$ can be expressed in the form of rational as $\frac{p}{q}$ (where p and q are coprime). The value of (p + q) is [Ans. 65]

[Sol. $f(x) = x + x\int_0^1 t f(t)\,dt + \int_0^1 t^2 f(t)\,dt$

$\therefore$ $f(x) = x(1 + A) + B$ where $A = \int_0^1 t f(t)\,dt$ and $B = \int_0^1 t^2 f(t)\,dt$

Now $A = \int_0^1 t[t(1+A)+B]dt = \left.\frac{t^3}{3}(1+A)\right|_0^1 + \left.\frac{B}{2}t^2\right|_0^1$

$A = \frac{1+A}{3} + \frac{B}{2} \Rightarrow 6A = 2(1 + A) + 3B \Rightarrow 4A - 3B = 2$ ....(1)

Again $B = \int_0^1 t^2[t(1+A)+B]dt = \left[\frac{t^4(1+A)}{4} + \frac{Bt^3}{3}\right]_0^1 = \frac{1+A}{4} + \frac{B}{3}$

12B = 3 + 3A + 4B $\Rightarrow$ 8B – 3A = 3 ....(2)

(1) × 3 gives $12A - 9B = 6$
(2) × 4 gives $-12A + 32B = 12$
adding

$$\Rightarrow \quad 23B = 18 \quad \Rightarrow \quad B = \frac{18}{23}$$

$$\Rightarrow \quad A = \frac{2+3B}{4} = \frac{25}{13}$$

$$\therefore \quad \int_0^1 f(x)\,dx = \int_0^1 \{(1+A)x + B\}\,dx$$

$$= \frac{1+A}{2} + B = \frac{1+A+2B}{2} = \frac{1+\frac{25}{23}+\frac{36}{23}}{2} = \frac{42}{23}$$

18 Let $I = \int_1^3 |(x-1)(x-2)(x-3)|\,dx$. The value of I^{-1} is [Ans. 2]

[Sol. $I = \int_1^3 |(x-1)(3-x)(x-2)|\,dx$

let $x = \cos^2\theta + 3\sin^2\theta$
$dx = 2\sin 2\theta\, d\theta$
$x - 1 = 2\sin^2\theta$; $3 - x = 2\cos^2\theta$ and $x - 2 = \cos^2\theta + 3\sin^2\theta - 2 = 2\sin^2\theta - 1 = -\cos 2\theta$

$$I = \int_0^{\pi/2} \left|2\sin\theta \cdot 2\cos^2\theta \cdot \cos 2\theta\right| 2\sin 2\theta\, d\theta = \int_0^{\pi/2} 4\sin^2\theta \cdot \cos^2\theta \cdot 2\sin 2\theta |\cos 2\theta|\, d\theta$$

$$= \int_0^{\pi/2} 2\sin^3 2\theta\, |\cos 2\theta|\, d\theta$$

put $2\theta = t$

$$I = \int_0^{\pi} 2\sin^3 t\, |\cos t| \frac{dt}{2} = 2\int_0^{\pi/2} (\sin^3 t \cdot \cos t)\, dt$$

put $\sin t = y$

$$I = 2\int_0^1 y^3\, dy = 2 \cdot \frac{y^4}{4}\Big|_0^1 = \frac{1}{2}$$

Exercise 2(A)

1 [Hint: $I = \int_1^\infty \frac{dx}{(e \cdot e^x + e^3 \cdot e^{-x})} = \int_1^\infty \frac{e^x\,dx}{e(e^{2x}+e^2)}$ (multiply N^r and D^r by e^x)

put $e^x = t \Rightarrow e^x\,dx = dt$

$I = \frac{1}{e}\int_e^\infty \frac{dt}{t^2+e^2} = \frac{1}{e^2}\tan^{-1}\frac{t}{e}\Big|_e^\infty = \frac{1}{e^2}\left[\frac{\pi}{2}-\frac{\pi}{4}\right] = \frac{\pi}{4e^2}$ Ans.]

2 [Hint: put $e^{x^2} = t$; $e^{x^2} \cdot 2x\,dx = dt$; $\int_1^{\pi/2}\cos t\,dt = [\sin t]_1^{\pi/2} = 1 - (\sin 1)$]

3 [Hint: Note that in $\left(-\frac{1}{2},\frac{1}{2}\right)$, $\sin^{-1}(3x-4x^3) = 3\sin^{-1}x$ and $\cos^{-1}(4x^3-3x) = 2\pi - 3\cos^{-1}x$

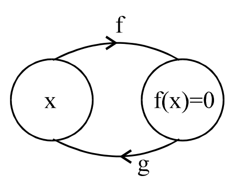
hence $f(x) = 3\sin^{-1}x - 2\pi + 3\cos^{-1}x = -\frac{\pi}{2}$

$\therefore I = -\frac{\pi}{2}\int_{-1/2}^{1/2} dx = -\frac{\pi}{2}$]

[Alternate: $f(x) = \sin^{-1}(3x-4x^3) - [\pi - \cos^{-1}(3x-4x^3)]$

$= -\pi + (\sin^{-1}(3x-4x^3) + \cos^{-1}(3x-4x^3)) = -\frac{\pi}{2}$]

4 [Sol. $f'(x) = \frac{1}{\sqrt{1+x^4}} = \frac{dy}{dx}$



now $g'(x) = \frac{dx}{dy} = \sqrt{1+x^4}$

when y = 0 i.e. $\int_2^x \frac{dt}{\sqrt{1+t^4}} = 0$ then x = 2 (think !)

hence $g'(0) = \sqrt{1+16} = \sqrt{17}$]

5 [Sol. $l = \ln \lim_{t\to 0} \frac{\int_0^t (1+a\sin bx)^{c/x}\,dx}{t} = \ln \lim_{t\to 0} (1+a\sin bt)^{c/t}$ (using L'Hospital's rule)

$= \ln e^{\lim_{t\to 0}\frac{c}{t}(a\sin bt)} = \lim_{t\to 0}\frac{abc\sin bt}{bt} = abc$ Ans.]

6 [Sol. $\sin nx - \sin(n-2)x = 2\cos(n-1)x\sin x$

$$\int\frac{\sin nx}{\sin x}dx = \int 2\cos(n-1)\,dx + \int\frac{\sin(n-2)x}{\sin x}dx$$

$$\therefore \int_0^{\pi/2}\frac{\sin 5x}{\sin x}dx = \int_0^{\pi/2}2\cos 4x\,dx + \int_0^{\pi/2}\frac{\sin 3x}{\sin x}dx = 0 + \int_0^{\pi/2}\frac{\sin 3x}{\sin x}dx = \int_0^{\pi/2}dx = \frac{\pi}{2}$$ Ans.]

7 [Sol. $F(x) = \frac{1}{2}\int\frac{(x^2+1)-(x-1)^2}{(x^2+1)(x-1)}dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\int\frac{x-1}{x^2+1}dx$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C$$

$\therefore$ discontinuous at $x = 1$

note that $f(x) = \int\frac{dx}{x^{1/3}} = \frac{3}{2}x^{2/3} + C$ is continuous although $\frac{1}{x^{1/3}}$ is discontinuous at $x = 0$]

8 [Sol. $T_r = \dfrac{1}{\sqrt{\frac{r}{n}}\cdot n\left(3\sqrt{\frac{r}{n}}+4\right)^2}$

$$S = \frac{1}{n}\sum_{1}^{4n}\frac{1}{\left(3\sqrt{\frac{r}{n}}+4\right)^2\cdot\sqrt{\frac{r}{n}}} = \int_0^4\frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

put $3\sqrt{x}+4 = t \Rightarrow \frac{3}{2}\frac{1}{\sqrt{x}}dx = dt$

$$= \frac{2}{3}\int_4^{10}\frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right]_{10}^{4} = \frac{2}{3}\left[\frac{1}{4}-\frac{1}{10}\right] = \frac{2}{3}\cdot\frac{6}{40} = \frac{1}{10}$$]

9 [Sol. $f'(x) = f(x) \Rightarrow f(x) = Ce^x$ and since $f(0) = 1$

$\therefore$ $1 = f(0) = C$ $\therefore$ $f(x) = e^x$ and hence $g(x) = x^2 - e^x$

Thus $\int_0^1 f(x)g(x)\,dx = \int_0^1 (x^2e^x - e^{2x})\,dx$

$$= x^2e^x\Big|_0^1 - 2\int_0^1 xe^x\,dx - \left[\frac{e^{2x}}{2}\right]_0^1 = (e-0) - 2\left[xe^x\Big|_0^1 - e^x\Big|_0^1\right] - \frac{1}{2}(e^2-1)$$

$= (e - 0) - 2\,[(e - 0) - (e - 1)] - \frac{1}{2}(e^2 - 1)$

$= e - \frac{1}{2}e^2 - \frac{3}{2}$]

10 [Sol. $I = \int_{-\pi/2}^{\pi/2} \frac{\cos\theta\, d\theta}{(2-\sin\theta)\cos\theta}$ (putting $x = \sin\theta$)

$= \int_0^{\pi/2} \left(\frac{1}{2-\sin\theta} + \frac{1}{2+\sin\theta}\right) d\theta \quad \left[\text{using} \int_{-a}^{a} f(x)\,dx = \int_0^{a} [f(x) + f(-x)\; dx\right]$

$= 4\int_0^{\pi/2} \frac{d\theta}{4-\sin^2\theta} = \frac{4}{3}\int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{\frac{4}{3} + \tan^2\theta} = \frac{4}{3}\int_0^{\infty} \frac{d\theta}{t^2 + \frac{4}{3}} = \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} \cdot \tan^{-1}\frac{\sqrt{3}\,t}{2}\Bigg|_0^{\infty} = \frac{2}{\sqrt{3}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$]

11 [Sol. $T_r = \frac{\pi}{6n}\sec^2\frac{r\pi}{6n}$

$S = \sum T_r = \frac{\pi}{6n}\sum_{r=1}^{n}\sec^2\frac{r\pi}{6n} = \frac{\pi}{6}\int_0^1 \sec^2\frac{\pi x}{6}\,dx = \tan\frac{\pi x}{6}\Bigg|_0^1 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$]

12 [Sol. Clearly f is an even function, hence

$I_1 = \int_0^{\pi} f(\cos(\pi - x))\,dx = \int_0^{\pi} f(-\cos x)\,dx = \int_0^{\pi} f(\cos x)\,dx$

$\therefore \quad I_1 = 2\int_0^{\pi/2} f(\cos x)\,dx = 2I_2 \quad \Rightarrow \quad \frac{I_1}{I_2} = 2$ **Ans.**

<u>Alternatively</u>: let $u = \cos x \quad \Rightarrow \quad du = -\sin x\,dx$

$\therefore \quad I_1 = \int_{-1}^{1} \frac{f(u)}{\sqrt{1-u^2}}\,du \quad \Rightarrow \quad 2\int_0^{1} \frac{f(u)}{\sqrt{1-u^2}}\,du$ ....(1)

|||ly with $\sin t = t$, $I_2 = \int_0^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt$ ....(2)

from (1) and (2) $\frac{I_1}{I_2} = 2$ Ans.]

13 [Hint: $\int_2^4 \left(\frac{\ln 2}{\ln x} - \frac{\ln 2}{\ln^2 x}\right) dx$ if $f(x) = \frac{1}{\ln x} \Rightarrow x f'(x) = -\frac{1}{\ln^2 x}$

$$\Rightarrow \quad I = \ln 2 \left(\frac{x}{\ln x}\right)_2^4 = \ln 2\left[\frac{4}{\ln 4} - \frac{2}{\ln 2}\right] = 0 \text{]}$$

14 [Hint: On rationalisation,

$$\int_{-1}^{1} \frac{(1+x^3)-\sqrt{1+x^6}}{1+x^6+2x^3-1-x^6}dx = \int_{-1}^{1}\frac{(1+x^3)-\sqrt{1+x^6}}{2x^3}dx = \underbrace{\frac{1}{2}\int_{-1}^{1}\frac{1}{x^3}dx}_{\text{odd} \Rightarrow \text{zero}} + \frac{1}{2}\int_{-1}^{1}dx - \underbrace{\int_{-1}^{1}\frac{\sqrt{1+x^6}}{2x^3}dx}_{\text{odd} \Rightarrow \text{zero}}$$

$$\frac{1}{2}\int_{-1}^{1}dx = \frac{1}{2}\cdot 2 = 1 \text{ Ans.]}$$

15 [Sol. at $y = 0$, $x = 2$

$f'(x) = \sqrt{9+x^4}\cdot 2x$

$$\therefore \quad g'(y) = \left.\frac{1}{f'(x)}\right|_{x=2} = \frac{1}{2x\sqrt{9+x^4}} = \frac{1}{20} \text{]}$$

16 [Sol. $\left.\dfrac{t^3}{3}\right|_0^{f(x)} = x\cos\pi x \quad \Rightarrow [f(x)]^3 = 3x\cos\pi x$ ....(1)

$$[f(9)]^3 = -27 \Rightarrow f(9) = -3$$

also differentiating $\displaystyle\int_0^{f(x)} t^2\,dt = x\cos\pi x$

$$[f(x)]^2 \cdot f'(x) = \cos\pi x - x\pi\sin\pi x$$

$\therefore \quad [f(9)]^2 \cdot f'(9) = -1$

$\Rightarrow \quad f'(9) = -\dfrac{1}{(f(9))^2} = -\dfrac{1}{9} \qquad f'(9) = -\dfrac{1}{9} \Rightarrow$ (A)]

17 [Hint: $\displaystyle\lim_{x\to\infty}\frac{x^{3/2}}{(x-1)}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2[1-(1/x)]} = \frac{1}{2}$ Ans.]

18 [Sol. $I = \displaystyle\int_1^e \underbrace{f''(x)}_{II}\,\underbrace{\ln x}_{I}\,dx = \ln x\cdot f'(x)\Big|_1^e - \int_1^e \frac{f'(x)}{x}dx$

$I = 1 - I_1$

$$I_1 = \int_1^e \frac{1}{x}f'(x)\,dx = \left.\frac{1}{x}\cdot f(x)\right|_1^e + \int_1^e\frac{f(x)}{x^2}dx$$

$$= \left(\frac{1}{e}-1\right) + \frac{1}{2}$$

$= \frac{1}{e} - \frac{1}{2}$

$\therefore \quad I = 1 - \frac{1}{e} + \frac{1}{2} = \frac{3}{2} - \frac{1}{e}$ Ans.]

19 [Sol. $f'(x)\frac{dy}{dx} = \frac{1}{\sqrt{x^4 + 3x^2 + 13}}$ when $y = f(x)$

$\therefore \quad g'(y) = \frac{1}{dy/dx} = \sqrt{x^4 + 3x^2 + 13}$

when y = 0 then x = 3

hence $g'(0) = \sqrt{3^4 + 27 + 13} = \sqrt{121} = 11$ Ans.]

20 [Hint: $I = \int \sqrt{1 + 2\,\text{cosec}\ x\ \cot x + 2\cot^2 x}$

$= \int \sqrt{\text{cosec}^2 x + 2\,\text{cosec}\,x \cot x + \cot^2 x}\, dx$

$= \int (\text{cosec}\,x + \cot x)\, dx$]

21 [Hint: $\left. \frac{t^2}{2} - \log_2 a \cdot t \right|_0^2 = 2 - \log_2(a^2)$

$(2 - 2\log_2 a) = 2 - 2\log_2 a$

$2\log_2 a = 2\log_2 a \quad \Rightarrow \quad a \in R^+$]

22 [Hint: Put $4x - 5 = 5t^2 \Rightarrow 4dx = 10t\, dt$ or better will be $5(4x - 5) = t^2$]

$$I = \frac{5}{2}\int_{\sqrt{\frac{3}{5}}}^{\sqrt{\frac{7}{5}}} \sqrt{\frac{5}{2}(1+t^2) - 5t} + \sqrt{\frac{5}{2}(1+t^2) + 5t}\ t\ dt = \left(\frac{5}{2}\right)^{3/2} \int_{\sqrt{\frac{3}{5}}}^{\sqrt{\frac{7}{5}}} \left(|t-1| + |(t+1)|\right) t\ dt$$

$$= \left(\frac{5}{2}\right)^{3/2} \left[\int_{\sqrt{\frac{3}{5}}}^{1} \left((1-t) + |(1+t)|\right) t\ dt + \int_{1}^{\sqrt{\frac{7}{5}}} \left((t-1) + (t+1)\right) t\ dt \right]$$

$$= \left(\frac{5}{2}\right)^{3/2} \left[2\int_{\sqrt{\frac{3}{5}}}^{1} t\ dt + \int_{1}^{\sqrt{\frac{7}{5}}} t^2\ dt \right]$$]

23 [Hint: $\frac{dy}{dx} = \frac{1}{\sqrt{y^2 + 1}}$

$$\frac{dy}{dx} = \sqrt{y^2+1}\,; \quad \frac{d^2y}{dx^2} = \frac{y}{\sqrt{y^2+1}}\sqrt{y^2+1} = y \text{ Ans.]}$$

24 [Hint: $f(x) = \sqrt{1+x^2} - x\,; \quad \lim_{x\to-\infty} x\left(\sqrt{1+x^2} - x\right) \to -\infty \Rightarrow$ DNE]

25 [Sol. $x = \frac{1}{t} \Rightarrow dx = \frac{1}{t^2}\,dt$

$$I = \int_2^{1/2} t\,\sin\left(\frac{1}{t} - t\right)\left(-\frac{1}{t^2}\right)dt = \int_2^{1/2} \frac{1}{t}\sin\left(t - \frac{1}{t}\right)dt = -\int_{1/2}^{2} \frac{1}{t}\sin\left(t - \frac{1}{t}\right)dt = -I$$

$\Rightarrow$ $2I = 0 \Rightarrow I = 0$

Alternatively : put $x = e^t \Rightarrow I = \int_{-\ln 2}^{\ln 2} \sin(e^t - e^{-t})dt = 0$ (odd function)]

26 [Sol. $f'(\ln x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ x & \text{for } x > 1 \end{cases}$

put $\ln x = t \Rightarrow x = e^t$

for $x > 1\,;\ f'(t) = e^t$ for $t > 0$

integrating $f(t) = e^t + C\,;\ f(0) = e^0 + C \Rightarrow C = -1$

$\therefore$ $f(t) = e^t - 1$ for $t > 0$ (corresponding to $x > 1$)

$\therefore$ $f(x) = e^x - 1$ for $x > 0$ ....(1)

again for $0 < x \le 1$

$f'(\ln x) = 1$ $(x = e^t)$

$f'(t) = 1$ for $t \le 0$

$f(t) = t + C$

$f(0) = 0 + C \Rightarrow C = 0 \Rightarrow f(t) = t$ for $t \le 0 \Rightarrow f(x) = x$ for $x \le 0$]

27 [Sol. $\int \frac{1}{x}\ln\frac{x}{e^x}dx = \int \frac{1}{x}(\ln x - \ln e^x)\,dx$

$$= \int \frac{\ln x - x}{x}dx = \left[\int \frac{1}{x}\ln x\,dx - \int \frac{1}{x}dx\right] \quad (\text{put } \ln x = u\,;\ \frac{1}{x}dx = du)$$

$$= \int u\,dx - \int 1\,dx = \frac{1}{2}\ln^2 x - x + C \quad]$$

28 [Sol. $\int_1^e e^x[x\ln x + 1 + \ln x - 1]dx = \int_1^e e^x[(\underbrace{x\ln x}_{f(x)}) + (\underbrace{\ln x + 1}_{f'(x)})]dx - \int_1^e e^x dx$

$$= e^x\cdot(x\ln x)\Big|_1^e - \left[e^x\right]_1^e = (e^e\cdot e - 0) - [e^e - e]$$

$= e^e(e-1) + e$ Ans.]

29 [Hint: $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}\right| \le \int_{10}^{19} \frac{|\sin x|}{1+x^8}\, dx \le \int_{10}^{19} \frac{dx}{1+x^8} < \int_{10}^{19} \frac{dx}{x^8} = \left[\frac{x^{-7}}{-7}\right]_{10}^{19}$

$= -\frac{1}{7}\,[19^{-7} - 10^{-7}] = \frac{1}{7}\,[10^{-7} - 19^{-7}] < 10^{-7}$]

30 [Sol. $\lim_{n\to\infty} \int_0^2 \left(1+\frac{t}{n+1}\right)^n dt = \lim_{n\to\infty} \left[\left(1+\frac{t}{n+1}\right)^{n+1}\right]_0^2 = \lim_{n\to\infty} \left(1+\frac{2}{n+1}\right)^{n+1} - 1 = e^2 - 1$

note that $\left[\left(1+\frac{t}{n+1}\right) \textit{is a linear function } a+bt \textit{ type}\right]$]

31 [Sol. $I = \int x\, 2^{\ln(x^2+1)} dx$ let $x^2 + 1 = t$; $x\,dx = \frac{dt}{2}$

Hence $I = \frac{1}{2}\int 2^{\ln t} dt = \frac{1}{2}\int t^{\ln 2} dt = \frac{1}{2}\cdot\frac{t^{\ln 2+1}}{\ln 2+1} + C = \frac{1}{2}\cdot\frac{(x^2+1)^{\ln 2+1}}{\ln 2+1} + C \Rightarrow$ (C)]

32 [Hint: $\int_0^1 (1+\cos^8 x)\, f(x)\, dx = \int_0^2 (1+\cos^8 x)\, f(x)\, dx =$

$\int_0^1 (1+\cos^8 x)\, f(x)\, dx + \int_1^2 (1+\cos^8 x)\, f(x)\, dx$

Hence $\int_1^2 (1+\cos^8 x)\, f(x)\, dx = 0$

$\Rightarrow (1+\cos^8 x)\, f(x) = 0$ at least once in (1,2)

but $1 + \cos^8 x \ne 0$

$\Rightarrow f(x) = ax^2 + bx + c$ vanishes at least once in (1,2)]

33 [Hint: $I = \int_0^{\pi/4} (1 - 2\sin^2 x)^{3/2} \cos x\, dx$. Put $\sqrt{2}\, \sin x = \sin\theta$

$\Rightarrow I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{16\sqrt{2}}$]

34 [Sol. Given $\int f(x)\,dx = g(x) \Rightarrow g'(x) = f(x)$

now $\frac{d}{dx}\left(\ln(1+g^2(x))\right) = \frac{2g(x)g'(x)}{1+g^2(x)} = \frac{2f(x)g(x)}{1+g^2(x)} \Rightarrow$ (B)]

35 [Sol. $\displaystyle \lim_{x\to 0} \frac{\int_0^x \sin t^2 dt}{x^3 \frac{(1-\cos x)}{x^2}}$ (using $\displaystyle \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$)

$$= \lim_{x\to 0} \frac{\int_0^x \sin t^2 dt}{x^3} \quad \text{(Using L'Hospital Rule)}$$

$$2 \lim_{x\to 0} \frac{\sin x^2}{3x^2} = \frac{2}{3} \quad \text{Ans.]}$$

36 [Sol. $\displaystyle I = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(-x)\,dx$ (using K)

$$2I = \int_{-1}^{1} \left(f(x) + f(-x)\right) dx = \int_{-1}^{1} (x^2)\,dx$$

$$2I = 2\int_{0}^{1} (x^2)\,dx \quad \Rightarrow \quad I = \int_{0}^{1} (x^2)\,dx = \frac{1}{3} \quad \text{Ans.]}$$

37 [Sol. $\displaystyle I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \frac{2x}{1+x^2}\,dx$ ....(1)

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{-2x}{1-x^4}\right) dx \quad \text{(using King)}$$

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \left(\pi - \cos^{-1} \frac{2x}{1-x^4}\right) dx \quad ....(2)$$

add (1) and (2)

$$\therefore \quad 2I = \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx$$

$$2I = 2\pi \int_{0}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx$$

$\therefore$ $k = \pi$ Ans.]

38 [Sol. $I = \int_0^{\pi/2} \sqrt{\tan x}\, dx$ ....(1) ; $I = \int_0^{\pi/2} \sqrt{\cot x}\, dx$ ....(2)

adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}} dx$$

$$= \sqrt{2}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = 2\sqrt{2}\int_0^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\,\pi \quad \text{(where } \sin x - \cos x = t)$$

$\therefore$ $I = \frac{\pi}{\sqrt{2}}$ Ans.]

39 [Hint: $I_1 = \int_{-\pi/4}^{\pi/4} \ln(\sin x + \cos x)\, dx = \int_{-\pi/4}^{\pi/4} \ln(\cos x - \sin x)\, dx$ (using king)

$\Rightarrow 2 I_1 = \int_{-\pi/4}^{\pi/4} \ln \cos 2x\, dx = 2\int_0^{\pi/4} \ln(\cos 2x) = \int_0^{\pi/2} \ln(\cos t)\, dt$ where $2x = t$

$\int_0^{\pi/2} \ln(\sin t)\, dt = I \Rightarrow I_1 = I/2$]

40 [Hint: $f'(x) = \frac{1}{x} + \pi\cos(\pi x) + C$

$f'(2) = \frac{1}{2} + \pi + C = \frac{1}{2} + \pi \quad \Rightarrow \quad C = 0$

$f(x) = \ln|x| + \sin(\pi x) + C'$

$f(1) = C' = 0$

$f(x) = \ln|x| + \sin(\pi x)$]

41 [Hint: $f'(x) = 1 + \ln^2 x + 2\ln x = 0 \Rightarrow (1 + \ln x)^2 = 0 \Rightarrow x = \frac{1}{e}$

Hence $f\left(\frac{1}{e}\right) = 1 + \frac{1}{e} + \int_1^{\frac{1}{e}} (\ln^2 t + 2\ln t)\, dt = 1 + \frac{1}{e} + t\ln^2 t\Big]_1^{\frac{1}{e}} = 1 + \frac{1}{e} + \frac{1}{e} = 1 + 2e^{-1} \Rightarrow$ [D]

42 [Sol. $I = \int_{-\infty}^{\infty} \underbrace{h'(x)}_{II} \cdot \underbrace{\sin x}_{I}\, dx = \sin x \cdot h(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \cos x \cdot h(x)\, dx = 0 - \cos 0 = -1 \Rightarrow$ (A)

note that here $\cos x = f(x)$]

43 [Sol. $I = \int_0^{\infty} (x^2)^n . x\, e^{-x^2}\, dx$ put $x^2 = t \Rightarrow x\, dx = -dt/2$

$$= \frac{1}{2}\int_0^{\infty} t^n \ e^{-t} \ dt \ = \frac{1}{2}\left[\ t^n \ e^{-t}\Big]_0^{\infty} \ + n\int_0^{\infty} t^{n-1} \ e^{-t} \ dt \ \right] = \frac{1}{2}\left[\ 0 + n\int_0^{\infty} t^{n-1} \ e^{-t} \ dt \ \right]$$

Hence $I = \frac{n!}{2}$]

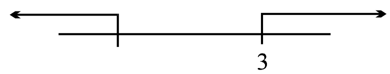
44 [Sol. $\int_a^0 3^{-x}\,(3^{-x} - 2)\ dx \geq 0$ put $3^{-x} = t \Rightarrow 3^{-x} \ln 3\ dx = -dt$

$$\ln 3 \int_1^{3^{-a}} (t-2)\ dt \ \geq 0 \quad \Rightarrow \quad \left.\frac{t^2}{2} - 2t\right]_1^{3^{-a}} \ \geq \ 0$$

$$\left(\frac{3^{-2a}}{2} - 2.3^{-a}\right) - \left(\frac{1}{2} - 2\right) \ \geq \ 0$$

$3^{-2a} - 4\times 3^{-a} + 3 > 0$

$(3^{-a} - 3)\,(3^{-a} - 1) > 0$



$3^{-a} > 3^1 \Rightarrow \ a < 1$

or $3^{-a} < 3^0 \ \Rightarrow \ a > 0$

Hence $a \in (-\infty, -1) \cup [0, \infty)$]

45 [Sol. $\sin(x + \alpha^2)\Big]_0^{\alpha} = \sin\alpha$

$\sin(\alpha^2 + \alpha) - \sin\alpha^2 = \sin\alpha$

$2 \cos(\alpha^2 + \alpha/2) \sin\alpha/2 = \sin\alpha$

now proceed and get

$\sqrt{2\pi} \ , \frac{-1 + \sqrt{1 + 8\pi}}{2} \ \Rightarrow$ 2 solutions]

46 Let $A = \int_0^1 \frac{e^t \ dt}{1 + t}$ then $\int_{a-1}^{a} \frac{e^{-t}\,dt}{t-a-1}$ has the value

(A) Ae^{-a} (B*) $-Ae^{-a}$ (C) $-ae^{-a}$ (D) Ae^{a}

[Hint : $I = \int_{a-1}^{a} \frac{e^{-t}}{t - a - 1}\ dt$ put t = a–1+ y (so that lower limit becomes zero)

$\therefore I = \int_0^1 \frac{e^{1-a-y}}{y-2}\ dy$ (now using king)

$$I = \int_0^1 \frac{e^{1-a-1+y}}{1-y-2}\ dy \ = -e^{-a}\int_0^1 \frac{e^y}{1+y}\ dy \ = -e^{-a} A \Rightarrow \text{(B)} \ ]$$

47 [Hint: $I = \int_0^1 \frac{e^t\,(t + 1 - t)}{(1+t)^2}\ dt \ = \int_0^1 \frac{e^t}{1+t}\ dt - \int_0^1 e^t \left(\frac{1}{1+t} - \frac{1}{(1+t)^2}\right)\ dt$

$$= A - \left.\frac{e^t}{1+t}\right]_0^1 = A - \frac{e}{2} + 1 \;;\; \text{Alternatively I. B. P. directly]}$$

48 [Hint : $\beta + \int_0^1 \underbrace{x}_{I}\underbrace{2xe^{-x^2}}_{II}dx = \int_0^1 e^{-x^2}\,dx$

$$\beta + \left[-xe^{-x^2}\right]_0^1 - \int_0^1 -e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx \qquad \beta = \frac{1}{e} \;]$$

49 [Sol. $g(x) = \int_0^x t\,\sin\frac{1}{t}\,dt$

$g'(x) = x\sin(1/x)$ which is diff $\Rightarrow$ g is cont. in $(0, \pi)$

$$l(x) = \begin{cases} x\sin x & 0 < x < \pi/2 \\ -\frac{\pi \sin x}{2} & \pi/2 < x < \pi \end{cases}$$

obvious discontinuity at $x = \pi/2 \Rightarrow$ (D)]

50 [Sol. $f(x) = \int_0^{\pi}\frac{t\sin t}{\sqrt{1+\tan^2 x\sin^2 t}}dt$

Using king and add.

$$f(x) = \frac{\pi}{2}\int_0^{\pi}\frac{\sin t}{\sqrt{1+\tan^2 x\sin^2 t}}dt \quad = \pi\int_0^{\pi/2}\frac{\sin t}{\sqrt{1+\tan^2 x(1-\cos^2 t)}}dt$$

$$= \pi\int_0^{\pi/2}\frac{\sin t}{\sqrt{\sec^2 x - \tan^2 x\cos^2 t}}dt \;=\; \pi\int_0^1\frac{dy}{\sqrt{\sec^2 x - \tan^2 x . y^2}}$$

$$= \frac{\pi}{\tan x}\int_0^1\frac{dy}{\sqrt{\operatorname{cosec}^2 x - y^2}} \quad = \frac{\pi}{\tan x}\left\{\sin^{-1}\frac{y}{\operatorname{cosec} x}\right\}_0^1 = \frac{\pi}{\tan x}\sin^{-1}(\sin x) = \frac{\pi x}{\tan x} \;]$$

51 [Sol. $I = \int_0^{n\pi+V}|\cos x|dx = \underbrace{\int_0^{n\pi}|\cos x|\,dx}_{2n} + \underbrace{\int_{n\pi}^{n\pi+V}|\cos x|\,dx}_{I_1\ (\text{put } x=n\pi+t)}$

$$\text{So, } I_1 = \int_0^V|\cos t|\,dt = \int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{V}\cos x\,dx$$

$= 1 - (\sin x)_{\pi/2}^{V} = 1 - \sin V + 1$

$\therefore \quad I = 2n + 2 - \sin V$]

52 [Sol. $\int \frac{p\,x^{p+2q-1} - qx^{q-1}}{(x^{p+q}+1)^2}\,dx = \int \frac{p\,x^{p-1} - qx^{-q-1}}{(x^{p}+x^{-q})^2}\,dx$

taking x^q as x^{2q} common from Denominator and take it in N^r]

53 [Hint: for $0 < x < \ln 2$, $[2e^{-x}] = 1$, otherwise zero $\Rightarrow I = \int_0^{\ell n 2} dx + \int_{\ell n 2}^{\infty} 0\,dx = \ln 2$

Alternatively: Put $e^{-x} = t$; $-x = \ln t$; $dx = -\frac{1}{t}dt$; Hence $I = -\int_1^0 \frac{[2t]dt}{t} = \int_0^1 \frac{[2t]dt}{t}$

$I = \int_0^{1/2} 0\,dt + \int_{1/2}^{1} \frac{dt}{t} = \ln t\Big]_{1/2}^{1} = 0 - \ln\frac{1}{2} = \ln 2$ Ans.]

54 [Sol. $2\int_0^1 \frac{dx}{\sqrt{x}} = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_0^1 = 4\left[\sqrt{x}\right]_0^1 = 4 \Rightarrow$ (C)]

55 [Sol. $I = \int_0^1 x\,\ln\left(\frac{x+2}{2}\right)dx = \int_0^1 x\,(\ln(x+2) - \ln 2)\;dx$

$\therefore \quad I = \int_0^1 x\,\ln(x+2)dx - \ln 2\int_0^1 x\,dx$; hence $I = \ln(x+2).\frac{x^2}{2}\Big]_0^1 - \int_0^1 \frac{x^2}{x+2}\,dx - \frac{\ln 2}{2}$

$= \frac{1}{2}\ln 3 - \int_0^1 \frac{x^2-4+4}{x+2}\,dx - \frac{\ln 2}{2} \Rightarrow \frac{1}{2}\ln\frac{3}{2} - \int_0^1\left((x-2) + \frac{4}{x+2}\right)dx$ now proceed]

56 [Sol. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)dx$; put $x = t^2$; $dx = 2t\,dt$

$= \int e^t(t^2+t)\,dt = e^t(At^2 + Bt + C)$ (Let)

Diffrentiate both the sides

$e^t(t^2 + t) = e^t(2At + B) + (At^2 + Bt + C)e^t$

On comparing coefficient we get

$A = 1$; $B = -1$; $C = 1$

57 [Hint: $I = \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1}\,dx + \int_{-1}^{1} \frac{|x|+1}{(|x|+1)^2}\,dx \Rightarrow 2\int_0^1 \frac{dx}{1+x} = 2\ln 2$]

odd $\Rightarrow$ vanishes even]

58 [Hint: Let $I = \int_0^{\pi/2} \frac{\sin x\,dx}{1+\sin x+\cos x}$

$I = \int_0^{\pi/2} \frac{\cos x}{1+\sin x+\cos x} \Rightarrow 2I = \int_0^{\pi/2} \frac{\sin x+\cos x+1-1}{\sin x+\cos x+1}\,dx$

$\Rightarrow 2I = \frac{\pi}{2} - \ln 2 \Rightarrow I = \frac{\pi}{4} - \frac{1}{2}\ln 2$]

59 [Sol. $\underset{x\to x_1}{\text{Limit}} \frac{\int_{x_1}^{x} f(t)\,dt}{\left(\frac{x-x_1}{x}\right)} = \underset{x\to x_1}{\text{Limit}} \frac{f(x).x^2}{x_1}$ (using Lopital's rule) $= x_1 f(x_1) \Rightarrow$ (B)]

60 [Sol. $I = \int_{-\pi/4}^{\pi/4} \ln(\cos x+\sin x)\,dx$

$I = \int_{-\pi/4}^{\pi/4} \ln(\cos x-\sin x)\,dx$ hence $2I = \int_{-\pi/4}^{\pi/4} \ln(\cos 2x)\,dx$

$= \int_0^{\pi/2} \cos t\,dt = -\frac{\pi}{2}\ln 2 \Rightarrow I = -\frac{\pi}{4}\ln 2$]

61 [Sol. $f(x) = \cos(\tan^{-1}x)$

$f'(x) = -\frac{\sin(\tan^{-1}x)}{1+x^2}$

$I = \int_0^1 x\,f''(x)\,dx = x\,f'(x)\Big]_0^1 - \int_0^1 f'(x)\,dx$

$= f'(1) - [f(x)]_0^1 = f'(1) - [f(1) - f(0)] = f'(1) - f(1) + f(0)$

$f(0) = 1$; $f'(1) = -\frac{1}{2\sqrt{2}}$; $f(1) = \frac{1}{\sqrt{2}}$]

62 [Hint: note that $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$ for $x > 0$

$$I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\left((\tan^{-1}x)^2 + 2\tan^{-1}x\right)dx \qquad \text{put } \tan^{-1}x = t$$

$$= \int e^t (t^2 + 2t)\,dt \;=\; e^t . t^2 \;=\; e^{\tan^{-1}x}\left(\tan^{-1}x\right)^2 + C \;]$$

63 [Hint: $I = \int_1^2 1.(ln x)^2\,dx \;=\; ln^2 x . x\Big]_1^2 - 2\int_1^2 \frac{ln x}{x}.x\,dx \;=\; 2\ln^2 2 - 2\left[\int_1^2 ln\,x\;dx\right]$

$= 2\ln^2 2 - 2\left[x\,ln x - x\right]_1^2 \;=\; 2\ln^2 2 - 2\,[\,(2\ln 2 - 2)\,(0 - 1)\,)\,]$

$= 2\ln^2 2 - 2\,[\,2\ln 2 - 1] = 2\ln^2 2 - 4\ln 2 + 2 \;=\; 2\,[\,\ln^2 2 - 2\ln 2 + 1\,] = 2\left(ln\frac{2}{e}\right)^2 \Rightarrow$ (B)]

64 [Sol. Given $U_n = \int_0^1 x^n.(2-x)^n\,dx$; $V_n = \int_0^1 x^n.(1-x)^n\,dx$

in U_n put $x = 2t \;\Rightarrow\; dx = 2dt$

$\therefore \quad U_n = 2\int_0^{1/2} 2^n.t^n\,2^n\,(1-t)^n\,dt \qquad$ ....(1)

Now $V_n = 2\int_0^{1/2} x^n\,(1-x)^n\,dx \qquad$ (Using Queen) .....(2)

From (1) and (2)

$U_n = 2^{2n}.V_n \Rightarrow$ (C)]

65 [Hint : $S'(x) = \ln x^3 . 3x^2 - \ln x^2 . 2x \qquad = 9x^2 \ln x - 4x \ln x$

$= x\,l n\,x\,(9x - 4)$. Hence $\frac{S'(x)}{x} = \ln x\,(9x - 4)$.

Now it is obvious that $\frac{S'(x)}{x}$ is continuous and derivable in its domain.]

66 [Hint : using L Hospital's rule

$$l = \lim_{x\to 0}\frac{-x\sin x}{2 - 2\cos 2x} = \lim_{x\to 0}\frac{-x\sin x}{2(2\sin^2 x)} = \lim_{x\to 0}\frac{-1}{4\frac{\sin x}{x}} = -\frac{1}{4} \quad]$$

67 [Hint : LHS = secx + cosecx $= 2\sqrt{2} \Rightarrow x = \frac{\pi}{4}$ and $\frac{11\pi}{12}$]

68 [Hint: $\underset{n\to\infty}{\text{Limit}} \dfrac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots\ldots\ldots+\sqrt{n}}{n\sqrt{n}} = \int_0^1 \sqrt{x}\,dx = \dfrac{2}{3}$

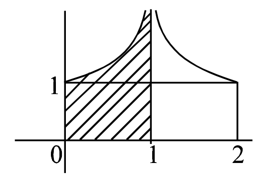
$\therefore \quad S_n = \dfrac{2}{3}n^{3/2}$]

69 [Sol. $\int_0^2 \dfrac{dx}{(1-x)^2} = \int_0^{1^-} \dfrac{dx}{(1-x)^2} + \int_{1^+}^2 \dfrac{dx}{(1-x)^2}$

$= \left.\dfrac{1}{1-x}\right]_0^{1^-} + \left.\dfrac{1}{1-x}\right]_{1^+}^2$

$= (\infty - 1) + (-1) - (-\infty) \Rightarrow$ indeterminant

Note that the shaded area is divergent]



70 [Hint: $I = \int_0^{\pi/2} \dfrac{\sin x \cos x}{x\left(\dfrac{\pi}{2}-x\right)}dx = \int_0^{\pi/2} \dfrac{\sin 2x}{x(\pi - 2x)}dx$; put $2x = t$

$I = \int_0^{\pi} \dfrac{\sin t}{t(\pi - t)}dt = \dfrac{1}{\pi}\int_0^{\pi}\left(\dfrac{\sin t}{t} + \dfrac{\sin t}{(\pi - t)}\right)dt = \dfrac{1}{\pi}\int_0^{\pi}\dfrac{\sin t}{t}dt + \dfrac{1}{\pi}\int_0^{\pi}\dfrac{\sin t}{\pi - t}dt$

$= \dfrac{1}{\pi}\int_0^{\pi}\dfrac{\sin t}{t}dt + \dfrac{1}{\pi}\int_0^{\pi}\dfrac{\sin t}{t}dt = \dfrac{2}{\pi}\int_0^{\pi}\dfrac{\sin t}{t}dt$ Ans.]

Q.1

Sol. $I = \int \frac{dx}{\cot\frac{x}{2}.\cot\frac{x}{3}.\cot\frac{x}{6}}$

$= \int \tan\frac{x}{2}.\tan\frac{x}{3}.\tan\frac{x}{6}\,dx$

$\because \frac{x}{2}-\frac{x}{3}=\frac{x}{6}$

$\tan\left(\frac{x}{2}-\frac{x}{3}\right)=\tan\frac{x}{6}$

$\tan\left(\frac{x}{2}-\frac{x}{3}\right)=\tan\frac{x}{6}$

or $\frac{\tan\frac{x}{2}-\tan\frac{x}{3}}{1+\tan\frac{x}{2}\tan\frac{x}{3}}=\tan\frac{x}{6}$

or $\boxed{\tan\frac{x}{2}-\tan\frac{x}{3}-\tan\frac{x}{6}=\tan\frac{x}{2}\tan\frac{x}{3}\tan\frac{x}{6}}$

or $I = \int\left(\tan\frac{x}{2}-\tan\frac{x}{3}-\tan\frac{x}{6}\right)dx$

$= \frac{\ell n\left(\sec\frac{x}{2}\right)}{\frac{1}{2}}-\frac{\ell n\left(\sec\frac{x}{3}\right)}{\frac{1}{3}}-\frac{\ell n\left(\sec\frac{x}{6}\right)}{\frac{1}{6}}+c$

or $\boxed{I = 2\ell n\left(\sec\frac{x}{2}\right)-3\ell n\left(\sec\frac{x}{3}\right)-6\ell n\left(\sec\frac{x}{6}\right)+c}$

$$= \int \tan\frac{x}{2} . \frac{\sec^2\frac{x}{2}}{\sqrt{\left(2-\sec^2\frac{x}{2}\right)\left(2+\sec^2\frac{x}{2}\right)}} dx$$

$$= \int \frac{\tan\frac{x}{2}\sec^2\frac{x}{2}}{\sqrt{4-\sec^4\frac{x}{2}}} dx$$

put $\boxed{\sec^2\frac{x}{2} = t}$

or $2\sec\frac{x}{2} \times \sec\frac{x}{2}\tan\frac{x}{2} \times \frac{1}{2} dx = dt$

or $\boxed{\sec^2\frac{x}{2}\tan\frac{x}{2} dx = dt}$

$$= \int \frac{dt}{\sqrt{4-t^2}}$$

$$= \sin^{-1}\left(\frac{t}{2}\right) + c$$

or $\boxed{I = \sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right) + c}$ **Ans**

Q.3

Sol. $\displaystyle\int \frac{\ell n\left(\ell n\left(\frac{1+x}{1-x}\right)\right)}{1-x^2} dx$

put $\ell n\left(\frac{1+x}{1-x}\right) = t$

$$\left(\frac{1-x}{1+x}\right) \times \frac{(1-x)-(1+x)(-1)}{(1-x)^2} dx = dt$$

$$= \int 1.dt + \int \frac{1}{t^2} dt$$

$$= t - \frac{1}{t} + c$$

or $\boxed{I = \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + c}$

Q.5

Sol. $I = \int \sqrt{\frac{\sin(x-a)}{\sin(x+a)}} dx$

$$= \int \sqrt{\frac{\sin(x-a) \times \sin(x-a)}{\sin(x+a)\sin(x-a)}} dx$$

$$= \int \frac{\sin(x-a)}{\sqrt{\sin^2 x - \sin^2 a}} dx$$

$$= \int \frac{\sin x \cos a}{\sqrt{1 - \cos^2 x - \sin^2 a}} dx - \int \frac{\cos x \sin a}{\sqrt{\sin^2 x - \sin^2 a}} dx$$

$$= \int \frac{\sin x \cos a}{\sqrt{(1 - \sin^2 a) - \cos^2 x}} - \int \frac{\cos x \sin a}{\sqrt{\sin^2 x - \sin^2 a}} dx$$

$I = I_1 - I_2$

$I_1 = \int \frac{\sin x \cos a}{\sqrt{\cos^2 a - \cos^2 x}} dx$

put $\cos x = u$

$-\sin x \, dx = du$

$= -\cos a \int \frac{du}{\sqrt{\cos^2 a - u^2}}$

$= -\cos a \sin^{-1}\left(\frac{\cos x}{\cos a}\right)$

$I_2 = \int \frac{\cos x \sin a}{\sqrt{\sin^2 x - \sin^2 a}} dx$

put $\sin x = v$

$\cos x \, dx = dv$

$= \int \frac{\sin a \, dv}{\sqrt{v^2 - \sin^2 a}}$

$= \sin a \, \ell n \left|\sin x + \sqrt{\sin^2 x - \sin^2 a}\right|$

$$= \int \frac{(t^3+1)}{t^3(t^2+1)} 6t^5 dt$$

$$= 6\int \frac{t^2(t^3+1)}{t^2+1} dt$$

$$= 6\int \frac{t^5+t^2}{t^2+1} dt$$

$$= 6\int \frac{(t^2+1)(t^3-t+1)+(t-1)}{t^2+1} dt$$

$$= 6\int \left[(t^3-t+1)+\frac{(t-1)}{t^2+1}\right] dt$$

$$= 6\int \left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}\int \frac{2t}{t^2+1} dt - \int \frac{1}{t^2+1} dt\right]$$

$$= \boxed{I = 6\left[\frac{t^4}{4}-\frac{t^2}{2}+t+\frac{1}{2}\ell n(t^2+1)-\tan^{-1} t\right]+c}$$

Q.8

Sol. $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

put $x = a\tan^2\theta$

$dx = a\, 2\tan\theta \sec^2\theta\, d\theta$

$$= \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\; a.2\tan\theta \sec^2 d\theta$$

$$= 2a\int \sin^{-1}(\sin\theta)\tan\theta.\sec^2\theta\, d\theta$$

$$= 2a\int \theta.\tan\theta.\sec^2\theta\, d\theta$$

$$= 2a\left[\theta.\int \tan\theta.\sec^2\theta d\theta - \int\left(1.\int \tan\theta \sec^2\theta.d\theta\right)d\theta\right]$$

$$= 2a\left[\theta\int t dt - \int (\int t.dt) d\theta\right]$$ put $\tan\theta = t$

$$= 2a\left[\theta.\frac{t^2}{2} - \int \frac{t^2}{2}.d\theta\right]$$ $\sec^2\theta\, d\theta = dt$

put $\ell nx = t = \frac{1}{x}dx = dt$

$$I = \int \tan t . \tan(t - \ell n2) \tan(\ell n2)\ dt$$

$$I = \int (\tan t - \tan(\ell n2) - \tan(t - \ell n2))dt$$

$= \ell n\sec t - t\tan(\ell n2) - \ell n\sec(t - \ell n2) + c$

$= \ell n\,(\sec(\ell nx)) - \ell n(x).\tan(\ell n2) - \ell n\left(\sec\left(\ell n\frac{x}{2}\right)\right) + c$

Q.11

Sol. $I = \int_1^2 \frac{(x^2-1)dx}{x^3\sqrt{2x^4-2x^2+1}}$

$$= \int_1^2 \frac{(x^2-1)\,dx}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}} = \int_1^2 \frac{(x^{-3}-x^{-5})}{\sqrt{2-2x^{-2}+x^{-4}}}\,dx$$

put $2 - 2x^{-2} + x^{-4} = t^2 \Rightarrow (x^{-3} - x^{-5})\,dx = \frac{1}{2}\ tdt$

when $x = 1$ then $t = 1$

&

$x = 2$ then $t = \frac{5}{4}$

$$I = \frac{1}{2}\int_1^{5/4}\frac{t}{t}dt = \frac{1}{2}\left(\frac{5}{4}-1\right) = \frac{1}{8}$$

$I = \frac{u}{v} = \frac{1}{8}$ then $\left(1000\left(\frac{1}{8}\right)\right) = 125$ Ans

Q.12

Sol. Given $\frac{d}{dx}(h(x)) = -\frac{\sin x}{\cos^2(\cos x)}$

$$= \int_0^{\pi/2} \sin^3 x dx + a^3 \int_0^{\pi/2} \cos^3 x dx + 3a \int_0^{\pi/2} \sin^2 x \cos x\, dx + 3a^2 \int_0^{\pi/2} \sin x \cos^2 x\, dx$$

$$= \frac{2}{3} + a^3 . \frac{2}{3} + 3a \int_0^{\pi/2} (1 - \cos^2 x) \cos x\, dx + 3a^2 \int_0^{\pi/2} (1 - \sin^2 x) \sin x\, dx$$

$$= \frac{2}{3} (1 + a^3) + 3a \left(1 - \frac{2}{3}\right) + 3a^2 \left(1 - \frac{2}{3}\right)$$

$$I_1 = \frac{2}{3} + \frac{2a^3}{3} + a + a^2$$

now

$$I_2 = \int_0^{\pi/2} x . \cos x\, dx \Rightarrow x. \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\; dx$$

$$I_2 = x \sin \Big|_0^{\pi/2} + \cos x \Big|_0^{\pi/2}$$

$$I_2 = \frac{\pi}{2} - 1$$

$$\text{therefore } I = I_1 - \frac{4a}{\pi - 2} . I_2$$

$$2 = \frac{2}{3} + \frac{2a^3}{3} + a + a^2 - \left(\frac{4a^2}{\pi - 2}\right)\left(\frac{\pi - 2}{2}\right)$$

$$2 = \frac{2}{3} + \frac{2a^3}{3} - a + a^2 \Rightarrow 2a^3 + 3a^2 - 3a + 2 = 6$$

$$2a^3 + 3a^2 - 3a - 4 = 0 \quad \begin{cases} a_1 \\ a_2 \\ a_3 \end{cases}$$

$$\text{so} \left\{ \begin{array}{c} a_1 + a_2 + a_3 = -\frac{3}{2} \\ a_1 a_2 + a_2 a_3 + a_3 a_1 = -\frac{3}{2} \end{array} \right\} \Rightarrow (a_1 + a_2 + a_3)^2 = a_1^2 + a_2^2 + a_3^2 + 2(a_1 a_2 + a_2 a_3 + a_3 a_2)$$

$a_1 a_2 a_3 = 2$ $\qquad \Rightarrow \frac{21}{4}$

Q.15

Sol. $$u = \int_0^{\pi/4} \left(\frac{\cos x}{\sin x + \cos x}\right)^2 dx$$

$$= 2\int_0^2 \frac{x^2}{\sqrt{x^2+4}}\,dx - 0 = 2\int_0^2 \frac{x^2+4-4}{\sqrt{x^2+4}}\,dx$$

$$= 2\int_0^2 \sqrt{x^2+4}\,dx - 8\int_0^2 \frac{1}{\sqrt{x^2+4}}\,dx$$

$$= 2\int_0^2 \sqrt{x^2+4}\,dx - 8\int_0^2 \frac{1}{\sqrt{x^2+4}}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\ell n(x+\sqrt{x^2+4})\right]_0^2 - 8\,\ell n\,(x+\sqrt{x^2+4})\Big|_0^2$$

$$= 4\sqrt{2} - 4\ell n(\sqrt{2}+1) \text{ Ans}$$

$$\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ell n\left(x+\sqrt{x+a^2}\right)+c$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ell n(x+\sqrt{x^2+a^2})+c$$

Q.18 $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}\,dx$

Sol. $\displaystyle I = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7-10x^5-7x^3+x}{x^2+2}\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^6-12x^2+1}{x^2+2}\,dx$

$$= 0 + 2\int_0^{\sqrt{2}} \frac{3x^6-12x^2+1}{x^2+2}\,dx$$

$$= 2\int_0^{\sqrt{2}} \frac{3x^2(x^4-4)+1}{x^2+2}\,dx = 2\int_0^{\sqrt{2}} \left(3x^2(x^2-2)+\frac{1}{x^2+2}\right)dx$$

$$= 2\int_0^{\sqrt{2}} \left(3x^4-6x^2+\frac{1}{x^2+2}\right)dx$$

$$= 2\left(\frac{3x^5}{5} - 2x^3 + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{2}\right)\right)_0^{\sqrt{2}}$$

Q.20 $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1} dx$

Sol. Put $x = \sin^2\theta \Rightarrow dx = \sin 2\theta \, d\theta$

$$I = \int_0^{\pi/2} \frac{(\theta \sin 2\theta)}{\sin^4\theta - \sin^2\theta + 1} d\theta \qquad ...(1)$$

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - \theta\right)\sin 2\theta}{\cos^4\theta - \cos^2\theta + 1} d\theta$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - \theta\right)\sin 2\theta}{(1-\sin^2\theta)^2 - (1-\sin^2\theta) + 1} d\theta = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - \theta\right)\sin 2\theta}{\sin^4\theta - \sin^2\theta + 1} d\theta \qquad ...(2)$$

(1) + (2) $\pi/2$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin 2\theta}{\sin^4\theta - \sin^2\theta + 1} d\theta$$

put $\sin^2\theta = t$

$$2I = \frac{\pi}{2} \int_0^1 \frac{dt}{t^2 - t + 1} = \frac{\pi}{2} \int_0^1 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$2I = \frac{\pi}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\left(t - \frac{1}{2}\right).2}{\sqrt{3}}\right)\Bigg|_0^1$$

$$2I = \frac{\pi}{\sqrt{3}}\left(\frac{2\pi}{6}\right) \Rightarrow I = \frac{\pi^2}{6\sqrt{3}}$$ **Ans**

$I = \frac{\pi}{8}\ \ell n2$ **Ans**

Q.22

Sol. Let $I = \int_{-\frac{1}{n}}^{1/n} (2007 \sin x)\,|x|\,dx + \int_{-\frac{1}{n}}^{1/n} (2008 \cos x)\,|x|\,dx$

odd vanish

$$I = \int_{-\frac{1}{n}}^{1/n} (2008 \cos x)\,|x|\,dx = 2\int_{0}^{1/n} ((2008)\cos x)x\,dx$$

$$= 2.\ 2008 \int_{0}^{1/n} x.\cos x\ dx$$

$$= 2.\ 2008 \left[x.\sin x\Big|_0^{1/n} - \int_{0}^{1/n} \sin x\,dx \right]$$

$$= 2.2008\left[\frac{1}{n}\sin\frac{1}{n} + \cos\frac{1}{n} - 1\right]$$

put $n = \frac{1}{y}$

$$= 2.2008\ \lim_{y\to\infty}\left[\frac{y\sin y + \cos y - 1}{y^2}\right]$$

$$= 2.2008\left[1 - \lim_{y\to 0}\frac{1-\cos y}{y^2}\right]$$

$= 2.2008\ .\ \frac{1}{2}\ = 2008$ **Ans**

$$= \int_0^{\pi} \sqrt{1+2(1+\cos 2x)+4\cos x}\ dx$$

$$= \int_0^{\pi} \sqrt{1+2.2\cos^2 x+4\cos x}\ dx$$

$$= \int_0^{\pi} \sqrt{4\cos^2 x+4\cos x+1}\ dx$$

$$= \int_0^{\pi} |2\cos x+1|\ dx = 2\sqrt{3}+\frac{5\pi}{3}$$

$$= \frac{\pi}{3/5}+\sqrt{12}$$

compare with $\left(\frac{\pi}{k}+\sqrt{w}\right)$ then

$k = \frac{3}{5}$; $w = 12$

so $k^2 + w^2 = \frac{9}{25}+144$

$$= \frac{3609}{25}$$ **Ans**

Q.25

Sol. $$= \int_0^1 \frac{(1-x^2)}{(1+x^2+2x)\sqrt{x+x^2+x^3}}\,dx$$

$$= \int_0^1 \frac{\left(1-\frac{1}{x^2}\right)}{\left(\frac{1}{x}+x+2\right)\sqrt{x+\frac{1}{x}+1}}\,dx$$

put $x + \frac{1}{x} + 1 = t^2$

$$I = \frac{a+b}{\sqrt{2}} \int_0^{\pi/2} dx \Rightarrow I = \frac{a+b}{2\sqrt{2}}\pi$$ **Ans**

Q.27

Sol. put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

when $x = 0 \Rightarrow \theta = 0$

& $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\sin^2\theta\, \ell n(\sin\theta)}{\cos\theta}.\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right).\ell n \sin\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \ell n \sin\theta\, d\theta - \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\ \ell n(\sin\theta) d\theta$$

$$= \frac{1}{2}\left(-\frac{\pi}{2}\ell n^2\right) - \frac{1}{2}\left[\ell n \sin\theta.\frac{\sin 2\theta}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta}.\frac{\sin 2\theta}{2} d\theta$$

$$= -\frac{\pi}{4}\ell n 2 + \frac{1}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{8}(1 - \ell n 4)$$ **Ans**

Q.28

Sol. $$I = \int_{\pi/4}^{\pi/3} \frac{(\sin^3\theta - \cos^3\theta - \cos^2\theta)}{\sin^2\theta\cos^2\theta}\left(\frac{\sin\theta + \cos\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)^{2007} d\theta$$

$$= \int_{\pi/4}^{\pi/3} (\tan\theta\sec\theta - \cot\theta\,\mathrm{cosec}\theta - \mathrm{cosec}^2\theta)\ (\sec\theta + \mathrm{cose}\theta + \cot\theta)^{2007} d\theta$$

put $\sec\theta + \mathrm{cosec}\theta + \cot\theta + t$

$(\sec\theta \tan\theta - \mathrm{cosec}\theta \cot\theta - \mathrm{cosec}^2\theta)\, d\theta = dt$

when $\theta = \pi/4$

$$= (\pi+3).2\int_0^{2/\pi} \frac{\sin x}{1+\cos^2 x}dx$$

put cos x = t $\Rightarrow$ – sinx dx = dt

where $\quad x = 0 \Rightarrow t = 1$

&

$$x = \frac{\pi}{2} \Rightarrow t = 0$$

$$= (\pi + 3) \cdot 2\int_0^1 \frac{dt}{1+t^2}$$

$$= (\pi+3)\tan^{-1} t\Big|_0^1 = (\pi + 3)\ \frac{\pi}{2}$$

Q.30

Sol. $I = \int_0^{\pi} \frac{(ax+b)\ \sec x \tan x}{\sec^2+3}\ dx \qquad ...(1)$

use prop $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$

$I = \int_0^{\pi} \frac{(a\pi - ax + b)\sec x \tan x}{\sec^2 x + 3}\ dx \qquad ...(2)$

(1) + (2)

$$2I = \int_0^{\pi} \frac{(a\pi + 2b)\sec x \tan x}{\sec^2 x + 3}\ dx$$

use prop $\int_0^{2\pi} f(x)dx = 2\int_0^{\pi} f(x)dx$

$$2I = 2(a\pi + 2b)\int_0^{\pi/2} \frac{\sec x \tan x}{\sec^2+3}dx$$

put sec x = t $\Rightarrow$ sec x tan x dx = dt

when $x = 0 \Rightarrow t = 1$ &

Exercise III

1 If the derivative of f(x) wrt x is $\frac{\cos x}{f(x)}$ then show that f(x) is a periodic function.

Sol Given $f'(x)=\frac{\cos x}{f(x)}$ $\Leftrightarrow f(x)\cdot f(x)=\cos x$

Integration both sides w.r.t.x $\left(f(x)\right)^2=\sin x+c$

$f(x)=\pm\sqrt{\sin x+c}$ where , ($c\in$ Real constant $n=\pm1$)

2 Find the range of the function, $f(x)=\int_{-1}^{1}\frac{\sin x\ dt}{1-2t\cos x+t^2}$.

Sol. $f(x)=\int_{-1}^{1}\frac{\sin x\ dt}{1-2t\cos x+t^2}$

$$=\sin x\int_{-1}^{1}\frac{1}{t^2-2t\cos x+\cos^2 x+1-\cos^2 x}dt$$

$$=\sin x\int_{-1}^{1}\frac{1}{(t-\cos x)^2+(\sin x)^2}dt$$

$$=\sin x\ \frac{1}{|\sin x|}\left[\tan^{-1}\left(\frac{t-\cos x}{\sin x}\right)\right]_{-1}^{1}$$

$$=\frac{\sin x}{|\sin x|}\left(\tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)-\tan^{-1}\left(\frac{-1-\cos x}{\sin x}\right)\right)$$

$$=\frac{\sin x}{|\sin x|}\left(\tan^{-1}\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)+\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)\right)$$

$$=\frac{\sin x}{|\sin x|}\left(\tan^{-1}\left(\tan\frac{x}{2}\right)+\tan^{-1}\left(\cot\frac{x}{2}\right)\right)$$

$$=\frac{\sin x}{|\sin x|}\left(\tan^{-1}\left(\tan\frac{x}{2}\right)+\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)\right)=\frac{\pi}{2}+\frac{\sin x}{|\sin x|}$$

3 A function f is defined in $[-1, 1]$ as $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$; $x \neq 0$; $f(0) = 0$; $f(1/\pi) = 0$. Discuss the continuity and derivability of f at $x = 0$.

Sol. $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}, x \neq 0$

$$f(x) = \int 2x\sin\frac{1}{x} - \cos\frac{1}{x}\,dx$$

$$= 2\int x\sin\frac{1}{x} - \int \cos\frac{1}{x}\,dx$$

$$= 2\left[\frac{x^2}{2}\sin\frac{1}{x} - \int \frac{x^2}{2}.\cos\frac{1}{x}.\left(\frac{-1}{x^2}\right)dx - \int \cos\frac{1}{x}\,dx\right.$$

$$= x^2\sin\frac{1}{x} + \int \cos\frac{1}{x}\,dx - \int \cos\frac{1}{x}\,dx + c$$

$$f(x) = x^2\sin\frac{1}{x} + c \qquad f\left(\frac{1}{\pi}\right) = \frac{1}{\pi^2}\sin\pi + c$$

$$c = 0$$

RHL $\lim_{x\to 0} f(x) = \lim_{x\to 0^+} x^2\sin\frac{1}{x} = 0$

LHL $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2\sin\frac{1}{x} = 0$

$f(0) = 0$

as f(x) is continuous at $x = 0$

RHD $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x^2\sin\frac{1}{x}-0}{x-0} = \lim_{x\to 0^+} x\sin\frac{1}{x} = 0$

LHD $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0^-}\frac{x^2\sin\frac{1}{x}-0}{x-0} = \lim_{x\to 0^-} x\sin\frac{1}{x} = 0$$

differentiable at $x = 0$

4 Let $f(x) = \begin{cases} -1 & \text{if } -2 \le x \le 0 \\ |x-1| & \text{if } 0 < x \le 2 \end{cases}$ and $g(x) = \int_{-2}^{x} f(t)\,dt$. Define g (x) as a function of x and test the continuity and differentiability of g(x) in (–2, 2).

Sol. $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ -(x-1), & 0 < x < 1 \\ (x-1), & 1 \le x \le 2 \end{cases}$

$$f(x) = \int_{-2}^{x} f(t)dt$$

Case I $-2 \le x \le 0$

$$f(x) = \int_{-2}^{x} -1dt = -[t]_{-2}^{x} = -(x+2)$$

Case II $0 < x < 1$

$$f(x) = \int_{-2}^{0} -1dt + \int_{0}^{x} -(t+1)dt$$

$$= -(0+2) - \left(\frac{t^2}{2} - t\right)_0^x$$

$$= -2 - \left(\frac{x^2}{2} - x\right) = -2 - \frac{x^2}{x} + x$$

Case II $1 \le x \le 2$

$$f(x) = \int_{-2}^{\infty} -1\,dt - \int_{0}^{1}(t-1)dt + \int_{1}^{x}(t-1)dt$$

$$= -(0+2) - \left(\frac{t^2}{2} - t\right)_0^1 + \left(\frac{t^2}{2} - t\right)_1^x$$

$$= -2 - \left(\frac{1}{2} - 1\right) + \frac{x^2}{2} - x - \left(\frac{1}{2} - 1\right)$$

$$-2 + \frac{1}{2} + \frac{x^2}{2} - x + \frac{1}{2}$$

$$= -1 + \frac{x^2}{2} - x$$

Now $$f(x) = \begin{cases} -(x+2) & -2 \le x \le 0 \\ -2 - \frac{x^2}{2} + x & 0 < x < 1 \\ -1 + \frac{x^2}{2} - x & 1 \le x \le 2 \end{cases}$$

checking continous at x = 0

LHL – (0 + 2) = – 2

RHL – 2 + 0 + 0 = – 2

continuous at x = 0

checking continouty at x = 1

LHL $-2 - \frac{1}{2} + 1 = -\frac{3}{2}$

RHL $= -1 + \frac{1}{2} - 1 = -\frac{3}{2}$

continuous at x = 1

$$f'(x) = \begin{cases} -1 & -2 \le x \le 0 \\ -x+1 & 0 < x < 1 \\ x-1 & 1 \le x \le 2 \end{cases}$$

$f'(0^{-1}) = -1$	$f'(1^{-}) = -1 + 1 = 0$
$f'(0^{+}) = 1$	$f'(1^{+}) = 1 - 1 = 0$

Not defferentiable at x = 0 & defferentiable at x = 1 **Ans.**

5 If $\phi(x) = \cos x - \int_0^x (x-t)\,\phi(t)\,dt$. Then find the value of $\phi''(x) + \phi(x)$.

Sol. $\phi(x) = \cos x - \int_0^x (x-t)\,\phi(t)\,dt$

$$\phi'(x) = -\sin - \left[\int_0^x \frac{d}{dx}(x-t)\phi(t)dt + (x-x)\phi(x)\frac{d}{dx}(x) - (0-t)\phi(0)\frac{d}{dx}(0)\right]$$

$$= -\sin x - \int_0^x \phi(t)dt$$

$f''(x) = -\cos x - \phi(x)$ so $\phi''(x) + \phi(x) = -\cos x$ **Ans.**

6 If $y = \frac{1}{a}\int_0^x f(t)\cdot \sin a(x-t)\,dt$ then prove that $\frac{d^2y}{dx^2} + a^2 y = f(x)$.

Sol. $y = \frac{1}{a}\int_0^x f(t)\cdot \sin a(x-t)\,dt$

$$\frac{dy}{dx} = \frac{1}{a}\left[\int_0^x \frac{d}{dx} f(t)\sin a(x-t)dt + f(x)\sin a(x-t)\frac{d}{dx}(x) - f(0)\sin a(x-0)\frac{d}{dx}(0)\right]$$

$$= \frac{1}{a}\int_0^x f(t)\cos a(x-t)(-a)\,dt$$

$$\frac{dy}{dx} = \int_0^x f(t)\cos a(x-t)dt$$

$$\frac{d^2y}{dx^2} = \left[\int_0^x \frac{d}{dx} f(t)\cos a(x-t)dt + f(x)\cos a(x-a)\frac{d}{dx}(x) - f(0)\cos a(x-0)\frac{d}{dx}(0)\right]$$

$$= \left[-a\int_0^x f(t)\sin a(x-t) + f(x)\right] = -a^2y + f(x)$$

$$\frac{d^2y}{dx^2} + a^2y = f(x)$$ **Ans.**

7 If $y = x^{\int_1^x \ln t\,dt}$, find $\frac{dy}{dx}$ at $x = e$.

Sol. $y = x^{\int_1^x \ln t\,dt}$

$= x^{[t\log t - t]_1^x}$

$= x^{(x\log x - x + 1)}$

$\log y = (x\log x - x + 1)\log x$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{x\log x - x + 1}{x}\right) + (\log x)(\log x + 1 - 1) = \frac{(x\log x - x + 1)}{x} + (\log x)^2$$

putting $x = e$

$$\frac{dy}{dx} = e^{(e\log e - e + 1)}\left[\frac{e\log e - e + 1}{e} + (\log e)^2\right] = e\left(\frac{1}{e} + 1\right) = (1 + e)$$ **Ans.**

8 A curve C_1 is defined by: $\frac{dy}{dx} = e^x \cos x$ for $x \in [0, 2\pi]$ and passes through the origin. Prove that the roots of the function (other than zero) occurs in the ranges $\frac{\pi}{2} < x < \pi$ and $\frac{3\pi}{2} < x < 2\pi$.

Sol. $\frac{dy}{dx} = e^x \cos x$, $\int dy = \int e^x \cos x \, dx$

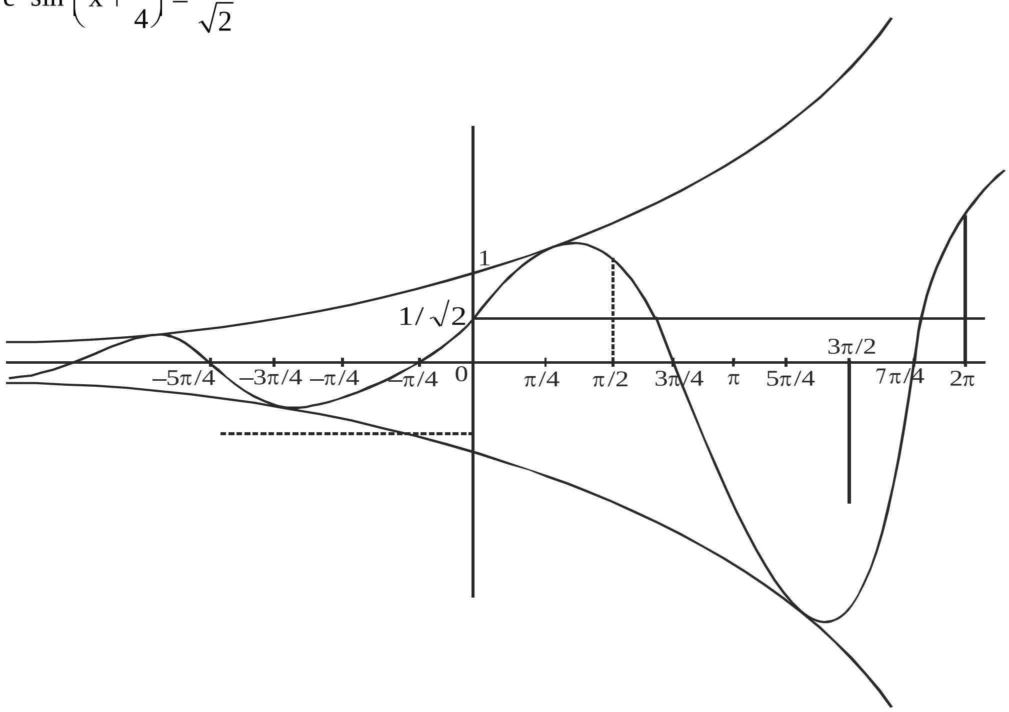
$y = \frac{e^x}{2}(\cos x + \sin x) + c$, putting x = 0, y = 0 , $0 = \frac{e^0}{2}(\cos 0 + \sin 0) + c$

$0 = \frac{1}{2}(1) + c$, $c = -\frac{1}{2}$, $y = \frac{e^x}{2}(\cos x + \sin x) - \frac{1}{2}$

putting y = 0

$\frac{e^x}{2}(\cos x + \sin x) - \frac{1}{2} = 0$, $e^x(\cos x + \sin x) = 1$

$e^x \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$



as one root lies between

$\frac{\pi}{2}$ & π & other lies bet $\frac{3\pi}{2}$ & 2π **Ans.**

9

(a) Let $g(x) = x^c \cdot e^{2x}$ & let $f(x) = \int_0^x e^{2t} \cdot (3t^2+1)^{1/2}\, dt$. For a certain value of 'c' , the limit of $\frac{f'(x)}{g'(x)}$ as $x \to \infty$ is finite and non zero . Determine the value of 'c' and the limit .

[Sol : $g'(x) = c\, x^{c-1} \cdot e^{2x} + x^c \cdot e^{2x} \cdot 2$

$f'(x) = e^{2x} (3x^2+1)^{1/2}$

$$\underset{x\to\infty}{\text{Limit}} \frac{f'(x)}{g'(x)} = \underset{x\to\infty}{\text{Limit}} \frac{e^{2x}\left(3x^2+1\right)^{1/2}}{c\, x^{c-1} \cdot e^{2x} + 2x^c \cdot e^{2x}} = \underset{x\to\infty}{\text{Limit}} \frac{x\left(3+\frac{1}{x^2}\right)^{1/2}}{x^c\left(\frac{c}{x}+2\right)}$$

If $x \to \infty$ it will be finite if $c = 1$ and $\underset{x\to\infty}{\text{Limit}}$ will be $\frac{\sqrt{3}}{2}$]

(b) Find the constants 'a' (a > 0) and 'b' such that , $\underset{x\to 0}{\text{Limit}} \frac{\int_0^x \frac{t^2\, dt}{\sqrt{a+t}}}{bx - \sin x} = 1$.

[Sol : $\frac{0}{0}$ form hence using L` Hospitals rule

$$l = \underset{x\to 0}{\text{Limit}} \frac{\frac{x^2}{\sqrt{a+x}}}{b - \cos x}$$ for existence of limit $\underset{x\to 0}{\text{Limit}}\; b - \cos x = 0$

$\Rightarrow$ b = 1

hence $\underset{x\to 0}{\text{Limit}} \frac{x^2}{1-\cos x} \cdot \underset{x\to 0}{\text{Limit}} \frac{1}{\sqrt{a+x}} = 1$ $\frac{2}{\sqrt{a}} = 1$

$\Rightarrow$ a = 4]

10 Evaluate: $\underset{x\to+\infty}{\text{Lim}} \frac{d}{dx} \int_{2\sin\frac{1}{x}}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)} dt$

[Sol. Use Leibniz's Rule. We know that

$$\frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}} f(t)\,dt = f\left(3\sqrt{x}\right)D\left(3\sqrt{x}\right) - f\left(2\sin\frac{1}{x}\right)D\left(2\sin\frac{1}{x}\right)$$

$$\frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)} dt = f\left(3\sqrt{x}\right)\frac{3}{2\sqrt{x}} + f\left(2\sin\frac{1}{x}\right)\frac{2}{x^2}\cos\frac{1}{x}$$

$$= \frac{3}{2}\,\frac{243x^2+1}{\sqrt{x}\left(3\sqrt{x}-3\right)(9x+3)} + 2\,\frac{\cos\left(\frac{1}{x}\right)\left(48\sin^4\left(\frac{1}{x}\right)+1\right)}{x^2\left(2\sin\left(\frac{1}{x}\right)-3\right)\left(4\sin^2\left(\frac{1}{x}\right)+3\right)}$$

simplifying and passing to the limit (using extended real number arithmetic) we find that the second term tends to 0 and so

$$\lim_{x\to+\infty}\frac{d}{dx}\int_{2\sin\frac{1}{x}}^{3\sqrt{x}}\frac{3t^4+1}{(t-3)(t^2+3)}dt=\frac{27}{2}=13.5\quad \text{Ans.}]$$

11 If $U_n=\int_0^{\pi/2}\frac{\sin^2 nx}{\sin^2 x}\,dx$, then show that $U_1, U_2, U_3, \ldots, U_n$ constitute an AP. Hence or otherwise find the value of U_n.

Sol $$U_n-U_{n-1}=\int_0^{\frac{\pi}{2}}\frac{\sin^2 nx-\sin^2(n-1)x}{\sin^2 x}\cdot dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{(\sin nx+\sin(n-1)x)(\sin nx-\sin(n-1))}{\sin^2 x}$$

$$=\int_0^{\frac{\pi}{2}}\frac{2\left(\sin\left(nx-\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)\left(2\sin\frac{x}{2}\cos\left(nx-\frac{x}{2}\right)\right)}{\sin^2 x}\cdot dx=\int_0^{\frac{\pi}{2}}\frac{\sin x-\sin((2n-1)x)}{\sin^2 x}\cdot dx$$

So,

$$U_n-U_{n-1}=\int_0^{\frac{\pi}{2}}\frac{\sin(2n-1)x}{\sin x}\cdot dx=f(n),\ \text{say}$$

$$f(n)-f(n-1)$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sin(2n-1)x-\sin(2n-3)x}{\sin x}\cdot dx=\int_0^{\frac{\pi}{2}}\frac{2\sin x\cos(2nx)}{\sin x}\cdot dx$$

$2nx=t \qquad \Rightarrow 2ndx=dt$

Hence,

$$f(n)-f(n-1)$$

$$=\underbrace{\int_0^{\pi}2\cos t\left(\frac{dt}{2n}\right)}_{0}$$

Hence,

$$f(n)=f(n-1)$$

i.e. U_n-U_{n-1} is a constant (Independent of n)

($U_n-U_{n-1}=U_{n-1}-U_{n-2}$, $\frac{U_n+U_{n-2}}{2}=U_{n-1}$ i.e. U_n, U_{n-1}, U_{n-2} are in A.P)

Hence, $U_1, U_2, \ldots, U_n$ caonstitutes an A.P

$$U_1 = \int_0^{\frac{\pi}{2}} dx\left(\frac{\pi}{2}\right) \quad , \quad U_2 = \int_0^{\frac{\pi}{2}} \frac{\sin^2 2x}{\sin^2 x}\cdot dx$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2 x\cdot dx = 2\int_0^{\frac{\pi}{2}}\left(\sin^2 x+\cos^2 x\right)\cdot dx \qquad \left(\int_a^b f = \int_a^b f\left(a+b-x\right)\right)$$

$$U_2 = \pi$$

$$U_2 - U_1 = \text{(common difference of A.P)} = \left(\frac{\pi}{2}\right)$$

Hence,

$$U_n = U_1 + (n-1)\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + (n-1)\frac{\pi}{2}$$

$$\boxed{U_n = n\frac{\pi}{1}}$$

12 If $\int_0^{\infty} \frac{\ell n\, t}{x^2+t^2}\, dt = \frac{\pi\ \ell n\, 2}{4}$ (x > 0) then show that there can be two integral values of ‘x’ satisfying this equation. [Ans: x = 2 or 4]

[Solution: put $t = x\tan\theta$

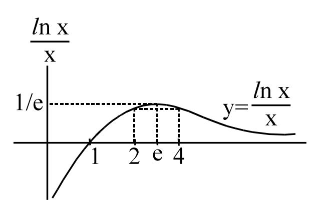
$$I = \int_0^{\pi/2} \frac{\ell n\,(x\tan\theta)\cdot x\sec^2\theta}{x^2(1+\tan^2\theta)}\, d\theta$$

$$= \frac{1}{x}\int_0^{\pi/2} (\ell n\, x + \ell n\tan\theta)\, d\theta$$

$$= \frac{\ell n\, x}{x}\int_0^{\pi/2} d\theta + \frac{1}{x}\int_0^{\pi/2} \ell n\tan\theta\, d\theta \qquad = \frac{\pi\ \ell n\, x}{2\,x} + \text{zero}$$

Hence $\frac{\pi\ \ell n\, x}{2\,x} = \frac{\pi\ \ell n\, 2}{4} \Rightarrow \frac{\ell n\, x}{x} = \frac{\ell n\, 2}{2} \Rightarrow x = 2$ or 4

Note from the graph of $y = \frac{\ell n\, x}{x}$



that for all values of $y = \frac{\ell n\, x}{x} \in \left(0, \frac{1}{e}\right)$,

there can be two values of x on either side

of x = e for which $\frac{\ell n\, x}{x}$ will have the same value.]

13 $\lim_{x\to 0}\left(\int_0^1 (by + a(1-y))^x\, dy\right)^{1/x}$ (where, $b \neq a$)

[Sol. Consider $I = \int_0^1 (by + a(1-y))^x\, dy$

$$= \int_0^1 (a + (b-a)y)^x\, dy = \left[\frac{(a+(b-a)y)^{x+1}}{(x+1)}\cdot\frac{1}{b-a}\right]_0^1$$

$$I = \frac{1}{(x+1)(b-a)}\left(b^{x+1} - a^{x+1}\right) = \frac{1}{(x+1)}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)$$

now $$L = \lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}\cdot\left(\frac{1}{(x+1)}\right)^{1/x} = \underbrace{\lim_{x\to 0}\left(\frac{1}{(x+1)}\right)^{1/x}}_{1^\infty}\cdot\underbrace{\lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}}_{1^\infty}$$

$$\left(\begin{array}{l}\lim_{x\to 0}(x+1)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}(x+1-1)} = e \\ \Rightarrow \quad \dfrac{1}{(x+1)^{1/x}} = \dfrac{1}{e}\end{array}\right)$$

$\therefore$ $$L = \frac{1}{e}\cdot\underbrace{\lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}}_{l}$$

now, $$l = e^{\lim_{x\to 0}\frac{1}{x}\left(\frac{b^{x+1}-a^{x+1}-b+a}{b-a}\right)}$$

$$= e^{\frac{1}{b-a}\lim_{x\to 0}\frac{b(b^x-1)-a(a^x-1)}{x}} = e^{\frac{1}{b-a}(b\ln b - a\ln a)} = e^{\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}} = \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

$\therefore$ $$L = \frac{1}{e}\cdot\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$ Ans.]

14 Let a, b are real number such that $a + b = 1$ then find the minimum value of the integral $\int_0^\pi (a\sin x + b\sin 2x)^2\, dx$. [Ans. $\pi/4$]

[Sol. Let $I = \int_0^\pi (a\sin x + b\sin 2x)^2\, dx$

$$I = \int_0^{\pi} (a \sin x - b \sin 2x)^2 \, dx$$

add $$2I = 2\int_0^{\pi} (a^2 \sin^2 x + b^2 \sin^2 2x) \, dx$$

$$I = 2\int_0^{\pi/2} (a^2 \sin^2 x) \, dx + 2\int_0^{\pi/2} (b^2 \sin^2 2x) \, dx = 2a^2 \frac{\pi}{4} + 2b^2 \underbrace{\int_0^{\pi/2} \sin^2 2x \, dx}_{J}$$

Let $J = \int_0^{\pi/2} \sin^2 2x \, dx$; put $2x = t$

$$= \frac{1}{2}\int_0^{\pi} \sin^2 t \, dt = \int_0^{\pi/2} \sin^2 t \, dt = \frac{\pi}{4}$$

hence $I = \frac{\pi a^2}{2} + \frac{\pi b^2}{2} = \frac{\pi}{2}(a^2 + b^2)$

$$I(a) = \frac{\pi}{2}[a^2 + (1 - a)^2] = \frac{\pi}{2}[2a^2 - 2a + 1] = \pi\left[a^2 - a + \frac{1}{2}\right] = \pi\left[\left(a - \frac{1}{2}\right)^2 + \frac{1}{4}\right]$$

$\therefore$ I(a) is minimum when $a = \frac{1}{2}$ and minimum value $= \frac{\pi}{4}$ **Ans.**]

15 Find a positive real valued continuously differentiable functions f on the real line such that for all x

$$f^2(x) = \int_0^x \left[(f(t))^2 + (f'(t))^2\right] dt + e^2$$

[Sol. differentiating both sides w.r.t. x

$$2 f(x) \cdot f'(x) = (f(x))^2 + (f'(x))^2$$

or $(f(x) - f'(x))^2 = 0 \Rightarrow f'(x) = f(x)$

(from the given relation $f(0) = e^2 \Rightarrow f(0) = e$ or $-e$ (to be rejected))

now $\frac{f'(x)}{f(x)} = 1 \Rightarrow \ln(f(x)) = x + C$; but $f(0) = e$

$\therefore$ $\ln(e) = C \Rightarrow C = 1$

$\therefore$ $\ln(f(x)) = x + 1 \Rightarrow f(x) = e^{x+1}$ Ans.]

16 Let f(x) be a continuously differentiable function then prove that,

$$\int_1^x [t] f'(t) \, dt = [x]. f(x) - \sum_{k=1}^{[x]} f(k)$$ where [·] denotes the greatest integer function and $x > 1$.

[Sol. $\int_1^2 f'(t)\,dt + 2\int_2^3 f'(t)\,dt + 3\int_3^4 f'(t)\,dt + \ldots\ldots + [x]\int_{[x]}^x f'(t)\,dt$

$= [f(t)]_1^2 + 2[f(t)]_2^3 + 3[f(t)]_3^4 + \ldots\ldots + [x]\,[f(t)]_{[x]}^x$

$= (f(2) - f(1)) + 2(f(3) - f(2)) + 3(f(4) - f(3)) + \ldots\ldots + [x](f(x) - f([x]))$

$= -(f(1) + f(2) + f(3) + \ldots\ldots + f([x])) + f(x).[x] = f(x).[x] - \sum_{k=1}^{[x]} f(k)$]

17 Let $F(x) = \int_{-1}^{x}\sqrt{4+t^2}\,dt$ and $G(x) = \int_{x}^{1}\sqrt{4+t^2}\,dt$ then compute the value of $(FG)'(0)$ where dash denotes the derivative. [Ans. zero]

[Sol. $F(x) = \int_{-1}^{x} f(t)\,dt$ and $G(x) = \int_{x}^{1} f(t)\,dt$ where $f(t) = \sqrt{4-t^2}$

now $H(x) = F(x)\cdot G(x)$

$H'(x) = F(x)\cdot G'(x) + G(x)\cdot F'(x)$

$$H'(x) = \left(\int_{-1}^{x} f(t)\,dt\right)\left(-\sqrt{4+x^2}\right) + \left(\int_{x}^{1} f(t)\,dt\right)\left(\sqrt{4+x^2}\right)$$

$$H'(x) = \sqrt{4+x^2}\left[\int_{x}^{1}\sqrt{4+t^2}\,dt - \int_{-1}^{x}\sqrt{4+t^2}\,dt\right];\ H'(0) = 2\left[\int_{0}^{1}\sqrt{4+t^2}\,dt - \int_{-1}^{0}\sqrt{4+t^2}\,dt\right]$$

put $t = -y$

$$= \left[\int_{0}^{1}\sqrt{4+t^2}\,dt + \int_{1}^{0}\sqrt{4+y^2}\,dy\right] = \left[\int_{0}^{0}\sqrt{4+t^2}\,dt\right] = \text{zero}$$ Ans.]

19 Evaluate:

(a) $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$;

(b) $\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n+1} + \frac{2}{n+2} + \ldots\ldots + \frac{3n}{4n}\right]$

Sol (a) Let,

$$S = \lim_{n\to\infty}\left(\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right)^{\frac{1}{n}} \qquad \because (S>0)$$

$$\Rightarrow \log S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \int_0^1 \log\left(1+x^2\right).dx$$ (Definite Integral as limit of sum.)

In tegrating by parts,

$$\Rightarrow \log S = \left(x\log\left(1+x^2\right)\right)_0^1 - \int_0^1 \frac{2x^2}{1+x^2}\cdot dx$$

$$\underset{(\log 2)}{\uparrow}$$

$$\Rightarrow \log\left(\frac{S}{2}\right) = 2\left[\int_0^1 \frac{dx}{1+x^2} - \int_0^1 .dx\right]$$

$$= 2\left[\frac{\pi}{4}-1\right] \qquad = \left(\frac{\pi-4}{2}\right)$$

$$\Rightarrow S = 2e^{\left(\frac{\pi-4}{2}\right)}.$$

(b) Let ,

$$S = \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n+1}+\frac{2}{n+2}+\cdots+\frac{3n}{4n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{3n}\frac{\left(\frac{r}{3n}\right)(3)}{1+\left(\frac{r}{3n}\right)(3)}$$

$$= 3\int_0^1 \frac{3x}{1+3x}\cdot dx$$ (Using Inegra as limit of sum)

$$= 3\int_0^1 1-\frac{1}{1+3x}\cdot dx \qquad = 3\left[1-\left(\frac{In(1+3x)}{3}\right)_0^1\right]$$

$$\boxed{S = (3 - In4)}.$$

20 Let $P_n = \sqrt[n]{\frac{(3n)!}{(2n)!}}$ (n = 1, 2, 3........) then find $\lim_{n\to\infty}\frac{P_n}{n}$.

Sol. $P_n = \left(\frac{(3n)!}{(2n)!}\right)^{1/n} = \left(\frac{(2n)!(2n+1)(2n+2)......(2n+n)}{(2n)!}\right)^{1/n}$

$$\therefore \quad \frac{P_n}{n} = \left(\frac{(2n+1)(2n+2)......(2n+n)}{n^n}\right)^{1/n} = \left(\frac{(2n+1)}{n}\cdot\frac{(2n+2)}{n}.........\frac{(2n+n)}{n}\right)^{1/n}$$

$$\therefore \quad ln\left(\frac{P_n}{n}\right) = \frac{1}{n}\sum_{r=1}^{n} ln\left(\frac{2n+r}{n}\right) = \frac{1}{n}\sum_{r=1}^{n} ln\left(2+\frac{r}{n}\right)$$

$$\therefore \quad \lim_{n\to\infty} \ln\left(\frac{P_n}{n}\right) = \int_0^1 \ln(2+x)\,dx = x\ln(2+x)\Big]_0^1 - \int_0^1 \frac{x}{x+2}\,dx$$

$$= \ln 3 - \left(\int_0^1 dx - \int_0^1 \frac{2\,dx}{x+2}\right) = \ln 3 - \left[1 - \left(2\ln(x+2)\right)_0^1\right]$$

$$= \ln 3 - \left[1 - (2\ln 3 - 2\ln 2)\right]$$

$$= \ln 3 - 1 + 2\ln 3 - 2\ln 2$$

$$= 3\ln 3 - 2\ln 2 - 1$$

$$= \ln\left(\frac{27}{4}\right) - \ln e = \ln\left(\frac{27}{4e}\right)$$

$$\therefore \quad \lim_{n\to\infty} \frac{P_n}{n} = \left(\frac{27}{4e}\right)$$ **Ans.**]

21 Let f be an injective function such that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)$ for all non negative real x & y with $f'(0) = 0$ & $f'(1) = 2 \neq f(0)$. Find f(x) & show that, $3\int f(x)\,dx - x\,(f(x)+2)$ is a constant.

Sol $g(x)3\int f(x).dx - x\left(f(x)+2\right)$

$\Rightarrow g'(x) = 3f(x) - f(x) - xf'(x) - 2$

$\Rightarrow g'(x) = 2f(x) - xf'(x) - 2$ ...(1)

$f(x).f(y) + 2 = f(x) + f(y) + f(xy)$

Partilly differentiating w.r.t.x,

$f'(x)\,f(y) = f'(x) + yf'(xy)$

Putting $x = 1$, $f'(1)\cdot f(y) = f'(1) + yf'(y)$

$\left(\because f'(1) = 2\right)$

Hence, $2f(y) - yf'(y) - 2 = 0$ $\left(\forall\, y \in R\right)$ ...(2)

Using (1) & (2), $g'(x) = 0$ $\left(\forall\, x \in R\right)$

Hence, $g(x)$ is a constant function.

22 Prove that $\sin x + \sin 3x + \sin 5x + ... + \sin(2k-1)x = \frac{\sin^2 kx}{\sin x}$, $k \in N$ and hence prove that, $\int_0^{\pi/2} \frac{\sin^2 kx}{\sin x}\,dx = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{2k-1}$.

Sol We know,

$e^{ix} = \cos x + i\sin x$

$$e^{i3x} = \cos 3x + i\sin 3x$$

$$e^{i(2k-1)x} = \cos(2k-1)x + i\sin(2k-1)x$$

Adding all,

$$\underbrace{e^{ix} + e^{i3x} + ... + e^{i(2r-1)x}}_{\left(\text{G.P with common ratio } e^{i2x}\right)} = \sum_{r=1}^{k}\cos(2r-1)x$$

$$\underbrace{+i\sum_{r=1}^{k}\sin(2r-1)x}_{\substack{\uparrow \\ \text{S,say}}}$$

Hence,

$$S = \frac{\left(e^{ix}\right)\left(\left(e^{i2x}\right)^k - 1\right)}{\left(e^{i2x}-1\right)} \quad = \frac{\left(e^{2kx}-1\right)}{\left(e^{ix}-e^{-ix}\right)}$$

$$= \frac{\left(e^{i(2kx)}-1\right)}{(2i\sin x)} \quad = (i)\left(\frac{1-e^{i(2kx)}}{2\sin x}\right)$$

$$\sum_{r=1}^{k}\sin(2r-1)x = \left(\frac{1-\cos 2kx}{2\sin x}\right)$$

Hence,

$$\frac{2\sin^2 kx}{2\sin x} = \left(\frac{\sin^2 kx}{\sin x}\right) = \sin x + \sin 3x + ... + \sin(2k-1)x = \left(\frac{\sin^2 kx}{\sin x}\right)$$

Now, $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 kx}{\sin x}\cdot dx = \int_0^{\frac{\pi}{2}}\left(\sin x + \sin 3x + ... + \sin(2k-1)x\right).dx$

$$= \left(\cos x + \frac{\cos 3x}{3} + ... + \frac{\cos(2k-1)x}{(2k-1)}\right)_{\frac{\pi}{2}}^{0} = \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... + \frac{1}{2k-1}\right).$$

24

Sol $\quad (1-x)^n = \sum_{k=0}^{n}(-1)^k\, {}^nC_k\, x^k$

$$\Leftrightarrow x^m(1-x)^n = \sum_{k=0}^{n}(-1)^k\, {}^nC_k\, x^{m+k}$$

Integrating the equation from 0 to 1, $\displaystyle\int_0^1 x^m(1-x)^n.dx = \sum_{k=0}^{n}\frac{(-1)^k\, {}^nC_k}{m+k+1}$ .....(1)

$|||^{ly}$, consideing $(1-x)^m$, $\displaystyle\int_0^1 x^n(1-x)^m_{dx} = \sum_{k=0}^{m}\frac{(-1)^k\, {}^mC_k}{n+k+1}$ .....(2)

But, $\int_0^1 x^n (1-x)^m .dx = \int_0^1 x^m (1-x)^n .dx$ ....(3)

$\left(\int_a^b f(x) = \int_a^b f(a+b-x).dx \right)$

(1),(2),(3)

$\Rightarrow \sum_{k=0}^{n} \frac{(-1)^k \, {}^nC_k}{m+k+1} = \sum_{k=0}^{m} \frac{(-1)^k \, {}^mC_k}{n+k+1}.$

25

[Sol.

(a) in (0, 1) $4 - x^2 - x^3 < 4 - x^2$

$\frac{1}{4 - x^2 - x^3} > \frac{1}{4 - x^2}$

$\therefore \quad \frac{1}{\sqrt{4 - x^2 - x^3}} > \frac{1}{\sqrt{4 - x^2}}$

$\therefore \quad \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} > \int_0^1 \frac{dx}{\sqrt{4 - x^2}} = \sin^{-1} \frac{x}{2} \Big]_0^1 = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} < I$

Again $4 - x^2 - x^3 > 4 - 2x^2$ in (0, 1)

$\frac{1}{\sqrt{4 - x^2 - x^3}} < \frac{1}{\sqrt{4 - 2x^2}}$

$I < \frac{1}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{2 - x^2}} = \frac{1}{\sqrt{2}} \sin^{-1} \frac{x}{\sqrt{2}} \Big]_0^1 = \frac{\pi}{4\sqrt{2}} = \frac{\pi\sqrt{2}}{8} \Rightarrow \frac{\pi}{6} < I < \frac{\pi\sqrt{2}}{8}$

(b) $I = \int_0^2 e^{x^2 - x} .dx$

Let, $f(x) = e^{x^2 - x}$

$f'(x) = \left(e^{x^2 - x}\right)(2x - 1)$

in $x \in (0, 2)$

$f_{\min} = f\left(\frac{1}{2}\right) = e^{-\frac{1}{4}}$

$f_{\max} = e^2 \max\{f(0), f(2)\}$

Hence, $\int_0^2 f_{\min} < I < \int_0^2 f_{\max}$

$\boxed{2e^{-\frac{1}{4}} < I < 2e^2}$.

(c) $I=\int_0^{2\pi}\frac{dx}{10+3\cos x}=2\int_0^{\pi}\frac{dx}{10+3\cos x}$

(cos x repeats it self i.e. takes same value again in $(\pi, 2\pi)$)

Let, $\frac{x}{2}=t$

$$I=4\int_0^{\frac{\pi}{2}}\frac{dt}{10+3\cos 2t}$$

$$=4\int_0^{\frac{\pi}{2}}\frac{dt}{10+3\left(\frac{1-\alpha^2}{1+\alpha^2}\right)} \quad (\alpha=\tan t)$$

$$=4\int_0^{\frac{\pi}{2}}\frac{(1+\alpha^2).dt}{(13+7\alpha^2)}$$

Let , $m=\tan t$

$dm=\sec^2 t=(1+\alpha^2)dt$

$$I=\frac{4}{7}\int_0^{\infty}\frac{dm}{\frac{13}{7}+m^2}=\left(\frac{4}{7}\right)\left(\sqrt{\frac{7}{13}}\right)\left(\tan^{-1}\left(\sqrt{\frac{7}{13}}m\right)\right)_0^{\infty}=\left(\frac{4}{\sqrt{91}}\right)\left(\frac{\pi}{2}\right)$$

$$\boxed{I=\left(\frac{2\pi}{\sqrt{91}}\right)}$$

$\sqrt{49}<\sqrt{91}<\sqrt{169}$ ($\sqrt{x}$ is function on $x>0$)

Hence, $\frac{2\pi}{13}<I<\frac{2\pi}{7}$.

(d) $I=\int_0^{2}\frac{dx}{2+x^2}=\frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{x}{\sqrt{2}}\right)_0^2 \quad =\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2})$

Let , $f(x)=\frac{1}{1+x^2}$

$$f'(x)=\frac{-2x}{(2+x^2)^2}<0 \quad \forall x>0 \text{ or } x\in(0,2),$$

$$f_{\min}=f(2)=\left(\frac{1}{6}\right),\ f_{\max}=f(0)=\left(\frac{1}{2}\right)$$

$$\int_0^2 f_{\min} < I < \int_0^2 f_{\max}$$
$$\underset{\frac{1}{3}}{\uparrow} \qquad \underset{1}{\uparrow} \quad .$$

26

Sol $f(x) = x + \int_0^1 xy^2 . f(y).dy + \int_0^1 x^2 y\, f(y).dy$

$$f(x) = x + x\left(\int_0^1 y^2 . f(y).dy\right)$$

$$+x^2\left(\int_0^1 y.f(y).dy\right) \qquad \ldots\ldots(1)$$

Hence, $f(x) = Ax^2 + Bx$ (A,B are real constant)

Using (1),

$$Ax^2 + Bx = x + x\int_0^1 y^2\left(Ay^2 + By\right).dy$$

$$+x^2\int_0^1 y\left(Ay^2 + By\right).dy$$

$$= x + x\left[\frac{A}{5} + \frac{B}{4}\right] + x^2\left[\frac{A}{4} + \frac{B}{3}\right]$$

$$\Leftrightarrow x^2\left[\frac{3A}{4} - \frac{B}{3}\right] + x\left[\frac{3B}{4} - \frac{A}{5} - 1\right] = 0 \qquad \ldots.(2)$$

(2) is true $\forall x \in R$

Hence, $\frac{3A}{4} - \frac{B}{3} = 0$ ...(i) and,

$$\frac{3B}{4} - \frac{A}{5} - 1 = 0 \qquad \ldots(ii)$$

Solving (i) , (ii) , we get,

$$A = \left(\frac{81}{119}\right)$$

$$B = \left(\frac{180}{119}\right)$$

Hence, $\boxed{f(x) = \frac{80x^2}{119} + \frac{180x}{119}}$.

Q.27

Sol. $I_1 = \int_{-1}^{1} \{x+1\}\{x^2+2\}+\{x^2+2\}\{x^3+4\}dx$

$= \int_{-1}^{1} \{x\}\{x^2\}+\{x^2\}\{x^3\}dx$ $\qquad \because \{x+I\} = \{x\}$

$= \int_{-1}^{1} (x-[x])(x^2-[x^2])+(x^2-[x^2])(x^3-[x^3])dx$

$\int_{-1}^{0} (x+1)(x^2)+(x^2)(x^3+1)\,dx + \int_{0}^{1} (x.x^2+x^2.x^3)\,dx$

$= \int_{-1}^{0} x^2(x^3+x+2)\,dx + \int_{0}^{1} (x^3+x^5)\,dx$

$= \int_{-1}^{0} (x^5+x^3+2x^2)\,dx + \int_{0}^{1} (x^3+x^5)dx = \left(\frac{x^6}{6}+\frac{x^4}{4}+\frac{2x^3}{3}\right)_{-1}^{0} + \left(\frac{x^4}{4}+\frac{x^5}{5}\right)_{0}^{1}$

$= \frac{2}{3}$

Q.28

Sol. $I = \int_{1}^{16} \tan^{-1}\left(\sqrt{\sqrt{x}-1}\right) dx$

$\sqrt{x} = \sec^2\theta$ $\qquad$ when $x = 1$ $\sec^2\theta = 1$

$\sec\theta = 1$

$x = \sec^4\theta$ $\qquad$ $\theta = 0$

$dx = 4\sec^2\theta.\sec\theta\tan\theta\,d\theta$ $\qquad$ $x = 16$ $\quad \sec^2\theta = \sqrt{16}$

$\sec^2\theta = 4$

$\sec\theta = 2 \Rightarrow \theta = \pi/3$

$I = \int_{0}^{\frac{\pi}{3}} \tan^{-1}\sqrt{\sec^2\theta-1}\ 4\sec^4\theta\tan\theta\,d\theta$

$$= 4\int_0^{\frac{\pi}{3}} \theta \sec^4\theta\ \tan\theta\ d\theta$$

$$= 4\int_0^{\frac{\pi}{3}} \theta\ \sec^3\theta(\sec\theta\tan\theta)d\theta$$

$$= 4\left(\left[\theta\frac{\sec^4\theta}{4}\right]_0^{\pi/3} - \int_0^{\frac{\pi}{3}}\frac{\sec^4\theta}{4}d\theta\right) \text{ (using by parts)}$$

$$= \left(\frac{\pi}{3}(2)^4\right) - \int_0^{\frac{\pi}{3}} \sec^4\theta\ d\theta$$

$$= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}} \sec^4\theta d\theta$$

$$= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}} \sec^4 d\theta$$

$$= \frac{16\pi}{3} - \int_0^{\frac{\pi}{3}} \sec^2\theta.(1+\tan^2\theta)d\theta$$

$$= \frac{16\pi}{3} - \left(\int_0^{\frac{\pi}{3}} \sec^2\theta + \int_0^{\frac{\pi}{3}} \sec^2\theta\tan^2\theta d\theta\right)$$

$$= \frac{16\pi}{3} - \left[\tan\theta + \frac{\tan^3\theta}{3}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{16\pi}{3} - \left(\sqrt{3} + \frac{3\sqrt{3}}{3}\right)$$

$= \frac{16\pi}{3} - 2\sqrt{3}$

Q.29

Sol. put 2x = t

dx = dt/2

$$= \int_0^{2\pi} \frac{dx}{2 + \frac{2\tan x}{1+\tan^2 x}} dx = \frac{1}{2}\int_0^{2\pi} \frac{\sec^2 x}{\tan^2 x + \tan x + 1} dx$$

Q.30

Sol. $I'(a) = \int_0^a \frac{x}{(1+ax)(1+x^2)} dx$

Q.31

Sol. $= \int_0^{\frac{\ell n3}{2}} \frac{e^x + 1}{e^{2x}+1} dx$

$$= \int_0^{\frac{\ell n3}{2}} \frac{e^x}{e^{2x}+1} dx + \int_0^{\frac{\ell n3}{2}} \frac{1}{e^{2x}+1} dx$$

$$= \int_0^{\frac{\ell n3}{2}} \frac{e^x}{e^{2x}+1} dx + \int_0^{\frac{\ell n3}{2}} \frac{1}{e^{2x}+1} dx$$

put $e^x = t$

$e^x\, dx = dt$

$$= \int_1^{\sqrt{3}} \frac{dt}{t^2+1} + \int_0^{\frac{\ell n3}{2}} \frac{e^{-2x}}{1+e^{-2X}} dx$$

put $1 + e^{-2x} = p - 2e^{-2x}\, dx = dp$

$$= \tan^{-1} t \int_1^{\sqrt{3}} \frac{1}{2}\int_2^{4/3} \frac{dt}{t}$$

$$= \frac{\pi}{3} - \frac{\pi}{4} - \frac{1}{2}\ell nt\Big|_2^{4/3}$$

$$= \frac{\pi}{12} - \frac{1}{2}\left(\ell n \frac{4}{3}\right)$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \ell n \frac{2}{3}\right]$$ **Ans**

Q.32

Sol. $$I = \int_0^{2\pi} \frac{x^2 \sin x}{8+\sin^2 x} dx$$

$$= \int_0^{2\pi} \frac{(2\pi - x)^2 \sin(2\pi - x)}{8+\sin^2(2\sin x)} dx$$

$$I = \int_0^{2\pi} \frac{(-x^2 + 4\pi x - 4\pi^2)\sin x}{8+\sin^2 x} dx \qquad ...(2)$$

$$2I = \int_0^{2\pi} \frac{(4\pi x - 4\pi^2)\sin x}{8+\sin^2 x} dx$$

$$= 4\pi \int_0^{2\pi} \frac{x \sin x}{8+\sin^2 x} dx - 4\pi^2 \int_0^{2\pi} \frac{\sin x}{8+\sin^2 x} dx$$

$$I = 2\pi \int_0^{2\pi} \frac{x \sin x}{8+\sin^2 x} dx - 2\pi^2 \int_0^{2\pi} \frac{\sin x}{8+1-\cos^2 x} dx$$

Let $I_1 = \int_0^{2\pi} \frac{x \sin x}{8+\sin^2 x} dx$

let $x = n + t$ $\qquad dx = dt$

$$I_1 = \int_{-\pi}^{\pi} \frac{(n+t)(-\sin t)}{8+\sin^2 t} dt$$

$$I_1 = -\pi \int_{-n}^{\pi} \frac{\sin t}{8+\sin^2 t} dt - \int_{-n}^{\pi} \frac{t \sin t}{8+\sin^2 t} dt$$

$$I_1 = -2\int_0^{\pi} \frac{t \sin t}{8+\sin^2 t} dt$$

$$I_1 = -2\int_0^{\pi} \frac{(\pi + t)\sin t}{8+\sin^2 t} dt$$

$$2I_1 = -\ 2\int_0^{\pi} \frac{\pi \sin t}{8+\sin^2 t} dt$$

$$2I_1 = -\ 2\pi \int_0^{\pi} \frac{\sin t}{8+\sin^2 t} dt$$

$$I_1 = -\ \pi \int_0^{\pi} \frac{\sin t}{8+\sin^2 t} dt$$

$$= -\ \pi.2 \int_0^{\frac{\pi}{2}} \frac{\sin t}{9-\cos^2 t} dt$$

$$= 2\pi\ \frac{1}{6}\left[\log\left(\frac{3+\cos t}{3-\cos t}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{3}\left(\ell \pi 1 - \log\left(\frac{4}{2}\right)\right)$$

$$= \left(-\frac{\pi}{3}\log 2\right)$$

$$I_2 = \int_0^{2\pi} \frac{\sin x}{9-\cos^2 x} dx$$

$$I_2 \int_0^{2\pi} \frac{\sin(2\pi - x)}{9-\cos^2(2\pi - x)} dx$$

$$I_2 = -\int_0^{2\pi} \frac{\sin x}{9-\cos^2 x} dx$$

$$I_2 = 0$$

so ultimate

$$I = 2\pi I_1 - 2\pi^2 I_2$$

$$= 2\pi\left(-\frac{\pi}{3}\log 2\right)$$

$$= -\frac{2\pi^2}{3}\log 2 \ \textbf{Ans.}$$

Q.33

Sol. $I = \frac{1}{2}\int_0^1 (2\sin\alpha x.\sin\beta x)\,dx \ = \frac{1}{2}\int_0^1 (\cos(\alpha-\beta)x - \cos(\alpha+\beta)x)\,dx$

$$= \frac{1}{2}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta}x - \frac{\sin(\alpha+\beta)x}{\alpha+\beta}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta} - \frac{\sin(\alpha+\beta)}{\alpha+\beta}\right]$$

Now

$$\left.\begin{matrix} 2\alpha = \tan\alpha \\ 2\beta = \tan\beta \end{matrix}\right\} \Rightarrow \begin{matrix} 2(\alpha-\beta) = \tan x - \tan\beta \\ 2(\alpha+\beta) = \tan x + \tan\beta \end{matrix}$$

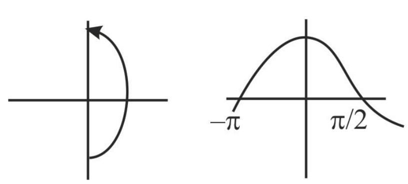
$$\therefore\ 2(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} \ \& \ 2(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

put these values

$I = \cos\alpha\cos\beta - \cos\alpha\cos\beta = 0$ **Ans**

Q.34

Sol.



$$= \int_0^p \cos x dx \ + \int_p^{p+q\pi} |\cos x|\, dx$$

$$= \sin x \int_0^p + q\int_0^{\pi} |\cos x|\, dx$$

$$= \sin p + q\left[\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right]$$

$= \sin p + q\ (1 + 1)$

$= 2q + \sin p$ **Ans**

Q.35.

Sol. $$f(\theta) \int_0^1 \frac{\tan^{-1} dx}{x^2 + 2x\cos\theta + 1}$$

$x = \tan\phi\ dx = \sec^2\phi\ d\phi$

$$= \int_0^{\frac{\pi}{2}} \frac{\phi \sec^2\phi\, d\phi}{\tan^2\phi + 2\tan\phi\cos\theta + 1}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\phi \sec^2\phi}{\sec^2\phi + 2\tan\phi\cos\theta} d\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{\phi.\frac{1}{\cos^2\phi}}{\frac{1}{\cos^2\phi} + \frac{2\sin\phi\cos\theta}{\cos\phi}} d\phi$$

$$= \int_0^{\frac{\pi}{2}} \frac{\phi}{1 + 2\sin\phi\cos\phi\cos\theta} d\phi$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\phi}{1 + (\sin 2\phi)\cos\theta} d\phi$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - \phi}{1 + (\sin 2\phi)\cos\theta} d\phi$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}}{1+(\sin 2\phi)\cos\theta} d\phi$$

$$I = \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{1}{1+(\sin 2\phi)\cos\theta} d\phi$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{2\tan\phi}{1+\tan^2\phi}\cos\theta} d\theta$$

$$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{\sec^2\phi}{\tan^2\phi + 2\tan\phi\cos\theta + 1} d\phi$$

let $\tan\phi = y$

$\sec^2\phi \; d\theta = dy$

$$= \frac{\pi}{4} \int_0^{\infty} \frac{dy}{y^2 + 2y\cos\theta + 1}$$

$$= \frac{\pi}{4} + \frac{\theta}{\sin\theta} = \frac{\pi\theta}{4\sin\theta}$$

Q.36

Sol. $$I = \int_0^{\pi} \frac{x\sin^3 x}{4-\cos^2 x} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin^3(\pi - x)dx}{4-\cos^2(\pi - x)}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin^3 x}{4-\cos^2(x)} dx$$

$$2I = \int_0^{\pi} \frac{\pi\sin^3 x}{4-\cos^2 x} dx$$

$$I = \frac{\pi}{2}.2\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{4-\cos^2 x}dx$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{4-\cos^2 x}dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{(1-\cos^2 x)\sin x}{(4-\cos^2 x)}dx$$

Let $\cos x = t$

$-\sin x\, dx = dt$

$$= -\pi\int_1^0 \frac{(1-t^2)}{(4-t^2)}dt$$

$$= \pi\int_0^1 \frac{(4-t^2)-3}{(4-t^2)}dt$$

$$= \pi\left[t-\frac{3}{2-t}\log\left(\frac{2+t}{2-t}\right)\right]_0^1$$

$$\pi\left(1-\frac{3}{4}\log 3\right)$$

$$= \pi\left(1-\frac{2\log b}{c}\right)$$

a = 3 b = 3

c = 4

= 3 + 3 + 4 = 10

Q.37

Sol. $I = \int_0^{\pi/2} \tan^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]dx$

$$\int_0^{\frac{\pi}{2}} \tan^{-1} \frac{\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|}{\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|}$$

$$= \int_0^{\frac{\pi}{2}} \tan^{-1}\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \tan^{-1}\left(\cot\frac{x}{2}\right)dx$$

$$I = \int_0^{\frac{\pi}{2}} \tan^{-1}\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)dx$$

$$0 < \frac{x}{2} < \frac{\pi}{4}$$

$$0 > -\frac{x}{2} > -\frac{\pi}{4}$$

$$\frac{\pi}{2} > \frac{\pi}{2}-\frac{x}{2} > \frac{\pi}{4}$$

$$\text{so } I = \int_0^{\frac{\pi}{2}} \frac{\pi}{2}-\frac{x}{2}\,dx$$

$$= \frac{\pi}{2}\times\frac{\pi}{2}-\frac{1}{4}\times\frac{\pi^2}{4}$$

$$= \frac{\pi^2}{4}-\frac{\pi^2}{16} = \frac{3\pi^2}{16}$$ **Ans.**

Q.38

Sol. $$\int_{\frac{\sqrt{3a^2+b^2}}{2}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x\,dx}{\sqrt{(x^2-a^2)\,(b^2-x^2)}}\,dx$$

Let $x^2 = a^2\cos^2\theta + b^2\sin^2\theta$ $2x\,dx = (-a^2\,2\cos\theta\sin\theta + b^2 2\sin\theta\cos\theta)\,d\theta$

$x^2 - a^2 = a^2\cos^2\theta + b^2\sin^2\theta - a^2$

$= b^2\sin^2\theta - a^2\sin^2\theta$

$= (b^2 - a^2)\sin^2\theta$

$= b^2 - b^2\cos^2\theta - a^2\sin^2\theta$

$= b^2\cos^2\theta - a^2\cos^2\theta$

$= (b^2 - a^2)\cos^2\theta$

$xdx = (b^2 - a^2)\sin 2\theta\, d\theta$

who $x^2 = \dfrac{3a^2 + b^2}{4}$

$0^2\cos^2\theta + b^2\sin^2\theta = \dfrac{3a^2 + b^2}{4}$

$4a^2\cos^2\theta + 4b^2\sin^2\theta = 3a^2 + b^2$

$4a^2 - 4a^2\sin^2\theta + 4b^2\sin^2\theta = 3a^2 + b^2$

$(a^2 - b^2) = 4(a^2 - b^2)\sin^2\theta$

$\sin^2\theta = 1/4$

$\sin\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{6}$

$x^2 = \dfrac{a^2 + b^2}{2}$

$a^2\cos^2\theta + b^2\sin^2\theta = \dfrac{a^2 + b^2}{2}$

$2a^2\cos^2\theta + 2b^2\sin^2\theta = a^2 + b^2$

$2a^2 - 2a^2\sin^2\theta + 2b^2\sin^2\theta = a^2 + b^2$

$(a^2 - b^2) = 2(a^2 - b^2)\sin^2\theta$

$\sin^2\theta = \dfrac{1}{2}$

$\sin\theta = \dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{\pi}{4}$

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(b^2 - a^2)\sin 2\theta d\theta}{\sqrt{(b^2 - a^2)\sin^2\theta(b^2 - a^2)}\cos^2\theta}$$

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(b^2 - a^2)\sin 2\theta\, d\theta}{(b^2 - a^2)\sin\theta\cos\theta}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta = \left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \left(\frac{3\pi - 2\pi}{12}\right) = \frac{\pi}{12}$$ **Ans.**

Q.39

Sol. $x^2 + 2x = k + \int_0^1 |t + k|\, dt$

CaseI, $k \geq 0$ $\quad |t + k| = \begin{cases} (t+k), (t \geq -k) \\ -(t+k),\ t \leq -k \end{cases}$

$$x^2 + 2x = k + \int_0^1 (t+k)\,dt$$

$$= k + \left[\frac{t^2}{2} + kt\right]_0^1$$

$$= k + \frac{1}{2}t + k = 2k + \frac{1}{2}$$

$2x^2 + 4x = 4k + 1$

$2x^2 + 4x - (4k + 1) = 0$

$D = (4)^2 + 4.2\,(4k + 1)$

$= 16 + 8(4k + 1)$

$= 8\,(2 + 4k + 1)$

$= 8\,(4k + 3)$

as $k \geq 0$ $\quad D > 0$

$\Rightarrow$ Roots are real & enequal.

Case II $k < 0$, let $k = -a$, $\quad a > 0$

$$x^2 + 2x = -a + \int_0^1 |t - a|\,dt$$

Now $|t - a| =$ $\quad (t - a), t > a$

$\quad -(t - a), t < a$

Now code $0 < a < 1$

$$x^2 + 2x = -a + \int_0^a -(t-a)dt + \int_a^1 (t-a)dt$$

$$= -a + \left(-\frac{t^2}{2} + at\right)_0^a + \left[\frac{t^2}{2} - at\right]_a^1$$

$$x^2 + 2x = -a + \left(-\frac{a^2}{2} + a^2 + \frac{1}{2} - a - \frac{a^2}{2} + a^2\right)$$

$$= -a + \left(2a^2 - a^2 - a + \frac{1}{2}\right)$$

$$= -a + a^2 - a + \frac{1}{2}$$

$x^2 + 2x = a^2 - 2a + \frac{1}{2}$

$2x^2 + 4x = 2a^2 - 4a + 1$

$2x^2 + 4x - (2a^2 - 4a + 1) = 0$

$D = 16 + 8(2a^2 - 4a + 1)$

$= 8(2 + 2a^2 - 4a + 1)$

$= 8\,(2a^2 - 4a + B)\ 16 - 4.2.3 < 0$

$\Rightarrow\ D > 0\ \ D \in (0,1)$

roots are real & unequal

$a \geq 1$

$x^2 + 2x = -a\ \int_0^1 (t-a)\,dt$

$$= -a - \left(\frac{t^2}{2} - at\right)_0^1$$

$$= -a - \left(\frac{1}{2} - a\right)$$

$$= -a + \frac{1}{2} + a$$

$$= -\frac{1}{2}$$

$2x^2 + 4x + 1 = 0$

$D > 16 - 8 \geq 0$

Roots one real & unequal

as real & unequal $\forall\ k \in B$

Q.40

Sol. $\int_{-1}^{1} \frac{2x^{332} + x^{998}}{1 + x^{666}}\,dx + \int_{-1}^{1} \frac{4x^{1668} \sin x^{691}}{1 + x^{666}}\,dx$

$= 2\int_0^1 \frac{x^{332}(1 + 1 + x^{666})}{1 + x^{666}}\,dx$

$$= 2\int_0^1 \frac{x^{332}}{1+x^{666}}\,dx + 2\int_0^1 x^{332}\,dx$$

$$= 2\int_0^1 \frac{x^{332}}{1+(x^{333})^2}\,dx + 2\int_0^1 x^{332}dx$$

Q.41

Sol. $$I = \int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

Let $x = \frac{\pi}{2} + t$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}+t\right)^2 \sin\left(\frac{2\pi}{2}+2t\right)\sin\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}+t\right)\right)}{2\left(\frac{\pi}{2}+t\right)-\pi}dt$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}+t\right)^2 \sin 2t \sin\left(\frac{\pi}{2}\sin t\right)}{t}dt$$

$$= \frac{1}{2}\left[\frac{\pi^2}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin 2x \sin\left(\frac{\pi}{2}+t\right)}{t}dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t^2 + 2t\sin\left(\frac{\pi}{2}\sin t\right)}{t}dt + \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin 2t + \sin\left(\frac{\pi}{2}\sin t\right)}{t}dt\right.$$

$$= \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2t \sin\left(\frac{\pi}{2}\sin t\right)dt$$

$$I = \frac{\pi}{2} - 2\int_0^{\frac{\pi}{2}} \sin 2t \sin\left(\frac{\pi}{2}\sin t\right)dt$$

$$\pi\int_0^{\frac{\pi}{2}} \sin 2t \sin\left(\frac{\pi}{2}\sin t\right)dt$$

Let $\sin t = y$

$\cos t\, dt = dy$

$$I = \pi\int_0^1 2y\sin\left(\frac{\pi}{2}y\right)dy$$

$$= 2\pi\int_0^1 y\sin\left(\frac{\pi}{2}y\right)dy$$

$$= 2\pi\left(\left[-\frac{y-\cos\frac{\pi}{2}y}{\frac{\pi}{2}}\right]_0^1 + \int_0^1 \frac{\cos\frac{\pi}{2}y}{\frac{\pi}{2}}dy\right)$$

$$= 2\pi\left[-\frac{y\cos\frac{\pi}{2}y}{\frac{\pi}{2}} + \frac{\sin\frac{\pi}{2}y}{\left(\frac{\pi}{2}\right)^2}\right]_0^1$$

$$= 2\pi\left(\frac{1}{\left(\frac{\pi}{2}\right)^2} - (0)\right)$$

$$- \ 2\pi \times \frac{1}{\pi^2} \times 4 = \frac{8}{\pi}$$

Q.42

Sol. $$\int_0^\infty \frac{dx}{x^2+2x\cos\theta+1}$$

$$\int_0^\infty \frac{dx}{x^2+2x\cos\theta+\cos^2\theta+1-\cos^2\theta}$$

$$\int_0^\infty \frac{dx}{(x+\cos\theta)^2+(\sin\theta)^2}$$

$$\frac{1}{\sin\theta}\left[\tan^{-1}\frac{x+\cos\theta}{\sin\theta}\right]_0^\infty$$

$$\frac{1}{\sin\theta}\left(\frac{\pi}{2}-\tan^{-1}\frac{\cos\theta}{\sin\theta}\right)=\frac{1}{\sin\theta}\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right)=\frac{\theta}{\sin\theta}$$

RHS $2\int_0^1\frac{dx}{x^2+2x\cos\theta+1}=\frac{2}{\sin\theta}.\left[\tan^{-1}\frac{\cos\theta}{\sin\theta}\right]_0^1$

$$=\frac{2}{\sin\theta}\left(\tan^{-1}\frac{1+\cos\theta}{\sin\theta}-\tan^{-1}\frac{\cos\theta}{\sin\theta}\right)$$

$$=\frac{2}{\sin\theta}\left(\tan^{-1}\cos\frac{\theta}{2}-\tan^{-1}\cos\theta\right)$$

$$=\frac{2}{\sin\theta}\left(\frac{\pi}{2}-\frac{\theta}{2}-\frac{\pi}{2}+\theta\right)=\frac{2}{\sin\theta}\left(\frac{\theta}{2}\right)=\frac{\theta}{\sin\theta}$$

LHS = RHS

Method II

$$\int_0^\infty\frac{dx}{x^2+2x\cos\theta+1}=\int_0^1\frac{dx}{x^2+2x\cos\theta+1}+\int_1^9\frac{dx}{x^2+2x\cos\theta+1}$$

Now $\int_0^\infty\frac{dx}{x^2+2x\cos\theta+1}$

$x=\frac{1}{t}$

$dx=-\frac{1}{t^2}dt$

$$\int_1^0\frac{\frac{-1}{t^2}dt}{\frac{1}{t^2}+\frac{2\cos\theta}{t}+1}$$

$$\int_1^0\frac{-1}{1+2t\cos\theta+t^2}=\int_0^1\frac{dt}{x^2+2t\cos\theta+1}$$

$$=\int_0^1\frac{dx}{x^2+2x\cos\theta+1}$$

so $\int_0^\infty \frac{dx}{x^2+2x\cos\theta+1} = 2\int_0^1 \frac{dx}{x^2+2x\cos\theta+1}$

Q.43

Sol. $I = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \left[k \int_k^{k+1} \sqrt{(x-k)\ (k+1-x)}\, dx \right]$

Now $I_1 = \int_k^{k+1} \sqrt{(x-k)((k+1)-x)}\, dx$

$x = k\cos^2\theta + (k+1)\sin^2\theta$

$dx = -2k\cos\theta\sin\theta + (kx)\ 2\sin\theta\cos\theta\, d\theta$

$= 2(k+1-k)\sin\theta\cos\theta\, d\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$x - k = k\cos^2\theta + (k+1)\sin^2\theta - k$

$= (k+1)\sin^2\theta - k(1-\cos^2\theta)$

$= (k+1-k)\sin^2\theta$

$= \sin^2\theta$

$(k+1) - x = (k+1) - k\cos^2\theta - (k+1)\sin^2\theta$

$= 1 + k\sin^2\theta - k\sin^2\theta - \sin^2\theta$

$= \cos^2\theta$

where $x = k$ $\quad k = k\cos^2\theta + (k+1)\sin^2\theta$

$k\sin^2\theta = (k+1)\sin^2\theta$

$k\sin^2\theta = k\sin^2\theta + \sin^2\theta$

$\sin^2\theta = 0$

$\theta = 0$

where

$x = k+1 \quad k+1 = k\cos^2\theta + (k+1)\sin^2\theta$

$(k+1)\cos^2\theta = k\cos^2\theta$

$\cos^2\theta = 0$

$\cos\theta = 0 \qquad \theta > \frac{\pi}{2}$

so $I_1 = \int_0^{\frac{\pi}{2}} (\sqrt{\sin^2\theta\ \cos^2\theta})\ 2\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta.2\sin\theta\cos\theta d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{\sin 4\theta}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$\text{so } I = \lim_{x\to\infty} \frac{1}{n^2} \sum_{k=0}^{n-1}\left[k\int_k^{k+1} \sqrt{(x-k)\,(k+1)-x)}dx\right]$$

$$= \lim_{x\to\infty} \frac{1}{n^2} \sum_{k=0}^{n-1}\left(k.\frac{n}{8}\right)$$

$$= \lim_{x\to\infty} \frac{1}{n} \sum_{k=0}^{n-1}\left(\frac{k}{n}\right)$$

$$= \frac{n}{8}\int_0^1 x dx = \frac{n}{8}\times\frac{1}{2} = \frac{n}{16}$$

Q.44

Sol. $\int_0^{\infty} f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{\ell nx}{x} dx$

$x = a\tan\theta$

$$I = \int_0^{\frac{n}{2}} f(\tan\theta + \cot\theta)\ \frac{\log(a\tan\theta)}{a\tan\theta}\ a\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{n}{2}} f(\tan\theta + \cot\theta)\ \log\ \frac{\tan(\theta)}{\frac{\sin\theta}{\cos\theta}} \frac{1}{\cos^2\theta} d\theta$$

$$I = 2\int_0^{\frac{\pi}{2}} f(\tan\theta + \cot\theta) \frac{\log(a\tan\theta)}{\sin 2\theta} d\theta \qquad ...(1)$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{f\left(\tan\left(\frac{\pi}{2}-\theta\right) + \cot\left(\frac{\pi}{2}-\theta\right)\log a\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)\right)}{\sin 2\left(\frac{\pi}{2}-\theta\right)} d\theta$$

$$I = 2\int_0^{\frac{\pi}{2}} \frac{f(\cot\theta + \tan\theta)\ \log(a\cot\theta)}{\sin 2\theta} d\theta \qquad ...(2)$$

(1) + (2)

$$2I = 2\int_0^{\frac{\pi}{2}} \frac{f(\tan\theta + \cot\theta)\log(a\tan\theta . a\cot\theta)}{\sin 2\theta} d\theta$$

$$I = \log a^2 \int_0^{\frac{n}{2}} \frac{f(\tan\theta + \cot\theta)}{\sin 2\theta} d\theta$$

$$= \log a \int_0^{\frac{\pi}{2}} \frac{f(\tan\theta + \cot\theta)}{\sin\theta\cos\theta} d\theta$$

after let $\tan\theta = \frac{x}{a}$ $\quad \sec^2\theta = \frac{1}{a}$ dx $\ d\theta = \frac{\cos^2\theta}{a} dx$

$$= \log a \int_0^{\infty} \frac{f\left(\frac{x}{a} + fa\right)x}{\sin\theta\cos\theta} \frac{\cos^2\theta}{a} dx$$

$$= \log a \int_0^{\infty} \frac{f\left(\left(\frac{x}{a}\right) + \frac{a}{x}\right)}{x} dx$$ **Proved.**

Q.45

Sol. $y = ax^2 + bx = c$

$y' = 2ax + b$

$y'(2) = 4a + b = 1$

$f(x) = ax^2 + (1 - 4a)x + c$

Now $\int_{2-\pi}^{2+\pi} f(x).\sin\left(\frac{x-2}{2}\right)dt$

let $x-2=t$

$dx = dt$

$$\int_{-x}^{x} f(x+2)\sin\left(\frac{t}{2}\right)dt$$

$\int_{-\pi}^{\pi}(a(t+2)^2+(1-4a)\ (t+2)+c)\ \sin\frac{t}{2}\,dt$

$= \int_{-\pi}^{\pi} at^2+\frac{t}{2}dt\ +\int_{-\pi}^{\pi} 4a\sin\frac{t}{2}dt\ + 4a\int_{-\pi}^{\pi} t\sin\frac{t}{2}dt\ +\ (1-4a)\int_{-\pi}^{\pi} t\sin\frac{t}{2}dt\ + 2(1-4a)$

$\int_{-\pi}^{\pi}\sin dt + c\int_{-\pi}^{\pi}\sin t\ dt$

$= (4a+1-4a)\int_{-\pi}^{\pi} t\sin\frac{t}{2}dt$

$= 2\int_{0}^{\pi} t\sin\frac{t}{2}dt$

$= 2\left[-2t\cos\frac{t}{2}+4\sin\frac{t}{2}\right]_0^{\pi}$

$= 2(4) = 8$

Q.46

Sol. $I = \int x\left(\sqrt{x+\frac{1}{x^2}}\right)\frac{\left[\ell n\,x^2+\ell n\left(1+\frac{1}{x^2}\right)-\ell nx^2\right]^2}{x^4}$

$$= \int\left[x\sqrt{1+\frac{1}{x^2}}\left[\frac{\ell nx^2+\ell n\left(1+\frac{1}{x^2}\right)-2\ell nx}{x^4}\right]dx\right]$$

$$= \int \left[\frac{1}{x^3} \sqrt{1+\frac{1}{x^2}} \left[\ell n \left(1+\frac{1}{x^2} \right) \right] \right] dx$$

put $1 + \frac{1}{x^2} = t$

$$\frac{-2}{x^3} dx = dt$$

$$= \left(\frac{-1}{2} \right) \int \sqrt{t} \ \ell nt \ dt$$

$$= \left(\frac{-1}{2} \right) \left[\int t^{1/2} . \ell nt \ dt \right]$$

$$= -\frac{1}{2} \int \underset{II}{t^{1/2}} . \underset{I}{\ell nt} \ dt \quad \text{(using by parts)}$$

$$= -\frac{1}{2} \left[\ell nt \int t^{1/2} dt - \int \frac{1}{t} \left(\int t^{1/2} dt \right) dt \right]$$

$$= -\frac{1}{2} \left[(\ell nt) \frac{t^{3/2}}{3/2} - \int \frac{t^{1/2}}{3/2} \ dt \right]$$

$$= -\frac{1}{3} t^{3/2} \ell nt - \frac{1}{3} \frac{t^{3/2}}{3/2} + c$$

$$\boxed{I = -\frac{1}{3} \left(1+\frac{1}{x^2} \right)^{3/2} \ell n \left(1+\frac{1}{x^2} \right) - \frac{2}{9} \left(1+\frac{1}{x^2} \right)^{3/2} + c}$$

Q.47

Sol. $I = \int \frac{\tan 2\theta}{\sqrt{\cos^6 \theta + \sin^6 \theta}} d\theta$

$$= \int \frac{\tan 2\theta}{\sqrt{\cos^4 \theta (1-\sin^2 \theta) + \sin^4 \theta (1-\cos^2 \theta)}} d\theta$$

$$= \int \frac{\tan 2\theta}{\sqrt{\cos^4\theta + \sin^4\theta - \sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)}} d\theta$$

$$= \int \frac{2\tan\theta.\sec^2\theta}{(1-\tan^2\theta)\sec^2\theta.\sqrt{\cos^4\theta + \sin^4\theta - \sin^2\theta\cos^2\theta}} d\theta$$

$$= \int \frac{dt}{(1-t)\sqrt{t^2 - t + 1}}$$ put $\tan^2\theta = t$

put $1 - t = \frac{1}{u}$ $\quad$ $2\tan\theta \sec^2\theta\, d\theta = dt$

$$\text{or } I = \int \frac{du}{u^2.\frac{1}{u}\sqrt{\left(1-\frac{1}{u}\right)^2 - \left(1-\frac{1}{u}\right) + 1}}$$

$$= \int \frac{du}{\frac{u}{u}\sqrt{u^2+1-u}} = \int \frac{du}{\sqrt{\left(u-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} = \ell n\left[\left(u-\frac{1}{2}\right) + \sqrt{\left(u-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \ell n\left[\left(\frac{1}{1-t}-\frac{1}{2}\right) + \sqrt{\left(\frac{1}{1-t}-\frac{1}{2}\right)^2 + \frac{3}{4}}\right] = \ell n\left[\left(\frac{1}{1-\tan^2\theta}-\frac{1}{2}\right) + \sqrt{\left(\frac{1}{1-\tan^2\theta}\frac{1}{2}\right)^2 + \frac{3}{4} + c}\right]$$

Q.48

Sol. $$I = \int \frac{\cot x\ dx}{(1-\sin x)\ (\sec x + 1)} = \int \frac{\frac{\cos x}{\sin x}}{(1-\sin x)\left(\frac{1}{\cos x}+1\right)} dx$$

$$= \int \frac{\cos x(1+\sin x)}{\sin x \cos^2 x\left(\frac{1+\cos x}{\cos x}\right)} dx$$

$$= \int \frac{1+\sin x}{\sin x(1+\cos x)} dx$$

$$= \int \frac{1+\sin x}{\sin x . 2\cos^2 \frac{x}{2}} dx$$

$$= \frac{1}{2}\int \operatorname{cosec} x . \sec^2 \frac{x}{2} dx + \frac{1}{2}\int \sec^2 \frac{x}{2} dx$$

$$= \frac{1}{2}\int \underset{u}{\operatorname{cosec} x} . \underset{v}{\sec^2 \frac{x}{2}} dx + \frac{1}{2}\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c$$

$$= \frac{1}{2}\left[\operatorname{cosec} x \int \sec^2 \frac{x}{2} dx - \int \left[(-\operatorname{cosec} x \cot x)\int \sec^2 \frac{x}{2} dx\right] dx\right] + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\left[\operatorname{cosec} x . \frac{\tan\frac{x}{2}}{\frac{1}{2}} + \int \operatorname{cosec} x \cot x . \frac{\tan\frac{x}{2}}{\frac{1}{2}} dx\right] + \tan\frac{x}{2} + c$$

$$= \operatorname{cosec} x . \tan\frac{x}{2} + \int \frac{\cos x}{\sin^2 x} \tan\frac{x}{2} dx + \tan\frac{x}{2} + c$$

$$= \frac{1}{\sin x}\tan\frac{x}{2} + \int \frac{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)^2} \tan\frac{x}{2} dx + \tan\frac{x}{2} + c$$

$$= \frac{\left(1+\tan^2\frac{x}{2}\right)}{2} + \int \frac{\left(1-\tan^2\frac{x}{2}\right)\left(1+\tan^2\frac{x}{2}\right)}{4\tan\frac{x}{2}} dx + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{4}\int \frac{\left(1-\tan^2\frac{x}{2}\right)}{\tan\frac{x}{2}}\sec^2\frac{x}{2}dx + \tan\frac{x}{2} + c$$

put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\int \frac{1-t^2}{t}dt + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\ell nt - \frac{1}{4}t^2 + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\ell n\left(\tan\frac{x}{2}\right) - \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + c$$

$$\frac{1}{2}\int \frac{1-t^2}{t}\,dt$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\left[\ell n\left|\tan\frac{x}{2}\right| - \frac{1}{2}\tan^2\frac{x}{2}\right] + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \frac{1}{2}\ell n\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\ell n\left|\tan\frac{x}{2}\right| + \tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2} - \frac{1}{4}\left(\sec^2\frac{x}{2} - 1\right) + c$$

$$\boxed{I = \frac{1}{2}\ell n\left(\tan\frac{x}{2}\right) + \tan\frac{x}{2} + \frac{1}{4}\sec^2\frac{x}{2} + c_1}$$

Q.49

Sol. $I = \int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx$

$$= \int \frac{e^x(1+1-x^2)}{(1-x)\sqrt{1-x^2}}dx$$

$$= \int e^x \left[\frac{1}{(1-x)\sqrt{1-x^2}} + \frac{1-x^2}{(1-x)\sqrt{1-x^2}} \right] dx$$

$$= \int e^x \left[\underbrace{\sqrt{\frac{1+x}{1-x}}}_{f(x)} + \underbrace{\frac{1}{(1-x)\sqrt{1-x^2}}}_{f'(x)} \right] dx$$

$$= e^x . f(x) + c$$

$$\boxed{I = e^x \sqrt{\frac{1+x}{1-x}} + c}$$ **Ans.**

Q.50

Sol. $I = \int \frac{3x^2+1}{(x^2-1)^3} dx$

$$= \int \frac{3x^2+1-x^2+x^2}{(x^2-1)^3} dx$$

$$= \int \frac{-(x^2-1)}{(x^2-1)^3} dx + \int \frac{4x^2}{(x^2-1)^3} dx$$

$$= \int \left[\frac{-1}{(x^2-1)^2} + x . \frac{4x}{(x^2-1)^3} \right] dx$$

This is the integral form of

$$\int [f(x) + xf'(x)] dx = xf(x) + c$$

$$= x\left(\frac{-1}{(x^2-1)^2} \right) + c$$

$$\boxed{I = -\frac{x}{(x^2-1)^2} + c}$$ **Ans.**

Q.51

Sol. $I = \int \frac{(ax^2 - b)\, dx}{x\sqrt{c^2x^2 - (ax^2 + b)^2}}$

diving by x^2

$$= \int \frac{\left(a - \frac{b}{x^2}\right) dx}{\sqrt{c^2 - \left(ax + \frac{b}{x}\right)^2}}$$

or $I = \int \frac{dt}{\sqrt{c^2 - t^2}}$ put $ax + \frac{b}{x} = t, \left(a - \frac{b}{x^2}\right) dx = dt$

$I = \sin^{-1}\left(\frac{t}{c}\right) + c$

$$\boxed{I = \sin^{-1}\left(\frac{ax + \frac{b}{x}}{c}\right) + c}$$

Q.52

Sol. $I = \int \frac{dx}{(x + \sqrt{x(1+x)})^2}\ dx$

$$= \int \frac{1}{x^2\left(1 + \sqrt{1 + \frac{1}{x}}\right)^2} dx$$

put $1 + \frac{1}{x} + t^2$

$-\frac{1}{x^2} dx = 2tdt$

or $I = \int \frac{-2t\ dt}{(1+t)^2} = -\int \frac{2t}{t^2 + 2t + 1}\ dt$

$$= -\left[\int \frac{2t+2}{t^2+2t+1}dt - \int \frac{2}{t^2+2t+1}\,dt\right.$$

$$= -\ell n\,(t+1)^2 - 2\int \frac{1}{(t+1)^2}dt$$

$$= -2\ell n\,(t+1) + \frac{2}{t+1} + c$$

or $\boxed{I = -2\ell n\left(1+\sqrt{1+\frac{1}{x}}\right) + \frac{2}{1+\sqrt{1+\frac{1}{x}}} + c}$

Q.53

Sol. $I = \int \frac{x+1}{x(1+xe^x)^2}dx$

$$= \int \frac{(x+1)e^x}{x.e^x(1+xe^x)^2}dx$$

or $I = \int \frac{1}{(t-1)t^2}\,dt$ put $1 + xe^x = t \Rightarrow (x.e^x + e^x.\,1)dx = dt \Rightarrow e^x(1 + x)dx = dt$

$$= \int \frac{(1-t^2)+t^2}{(t-1)t^2}\,dt$$

$$= \int \frac{-(1+t)}{t^2}\,dt + \int \frac{1}{(t-1)}dt$$

$$= -\int \frac{1}{t^2}\,dt - \int \frac{1}{t}dt + \int \frac{1}{(t-1)}dt$$

$$I = \frac{1}{t} - \ell n\,(t) + \ell n\,(t-1) + c$$

$$= \frac{1}{1+xe^x} - \ln(1 + xe^x) + \ell n(xe^x) + c$$

$$\boxed{I=\frac{1}{1+xe^{x}}+\ell n\left(\frac{xe^{x}}{1+xe^{x}}\right)+c}$$

Q.54

Sol. Let $f(x) = ax^2 + bx + 1$

$$I = \int \frac{f(x)dx}{x^2(x+1)^3}$$

$$= \int \frac{ax^2+bx+1}{x^2(x+1)^3}\,dx$$

or $\dfrac{ax^2+bx+1}{x^2(x+1)^3}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{(x+1)}+\dfrac{D}{(x+1)^2}+\dfrac{E}{(x+1)^3}$

Q.55

Sol. $f'(x)=\dfrac{1}{1+x^2}+\dfrac{1}{2}\left(\dfrac{1}{1+x}-\dfrac{-1}{1-x}\right)=\dfrac{2}{1-x^4}$

$$\int\frac{1}{2}f'(x)\,d\left(x^4\right)=\int\frac{1}{2}\cdot\frac{2}{1-x^4}.4x^3dx=\int\frac{-1}{t}dt$$

(where $t=1-x^4, dt=-4x^3dx$)

$$=-\ell n\,t+c=-\ell n\left|1-x^4\right|+c$$

Q.56

Sol. $= \displaystyle\int\frac{x(1+x)}{e^{2x}\left(1+\frac{x}{e^x}+\frac{1}{e^x}\right)^2}dx$

$$= \int \frac{x(1+x)e^{-2x}}{[1+(1+x)e^{-x}]^2} dx$$

or $I = \int \frac{x(1+x)e^{-x}.xe^{-x}}{(1+(1+x)e^{-x})^2} dx$

put $1 + (x + 1)e^{-x} = t$

$[0 + e^{-x}\, 1 + x\,(-e^{-x}) + e^{-x}\,(-1)]dx = dt$

$-xe^{-x}\, dx = dt$

or $I = -\int \frac{(t-1)}{t^2} dt$

$$= \int \frac{1}{t^2} dt - \int \frac{1}{t} dt$$

or $I = -\frac{1}{t} - \ell n t + c$

or $\boxed{I = -\frac{1}{1+(1+x)e^{-x}} - \ell n\,|1+(1+x)e^{-x}| + c}$

Q.57

Sol. $I = \int \frac{e^{\cos x}(x\sin^3 x + \cos x)}{\sin^2 x} dx$

$$= \int e^{\cos x}(x\sin x + \cot x \operatorname{cosec} x)\, dx$$

or $I = \int \underset{I}{x}.e^{\cos x} \underset{II}{\sin x}\, dx + \int \underset{I}{e^{\cos x}} \underset{II}{\operatorname{cosec}} \cot x\, dx$

$I = I_1 + I_2$

$I_1 = \int \underset{I}{x}\; \underset{II}{e^{\cos x}.\sin x}\, dx \quad = x\int e^{\cos x}\sin x dx - \int 1.\left(\int e^{\cos x}\sin x\, dx\right) dx$

$$= -xe^{\cos x} + \int 1.e^{\cos x} dx + c$$

$I_2 = \int \frac{e^{\cos x}}{I} . \frac{\operatorname{cosec} x \cot x}{II} dx$

$$= e^{\cos x}\int \operatorname{cosec} x \cot x\, dx - \int (e^{\cos x}(-\sin x)\int \operatorname{cosec} x \cot x\, dx)\, dx$$

$$= e^{\cos x}(-\operatorname{cosec} x) + \int e^{\cos x}.\sin x(-\operatorname{cosec} x)dx$$

$$= -e^{\cos x} \operatorname{cosec} x - \int e^{\cos x} dx + c$$

$$\therefore\ I = I_1 + I_2 = -xe^{\cos x} + \int e^{\cos x}dx - e^{\cos x}\operatorname{cosec} x - \int e^{\cos x}dx + c$$

$$= -e^{\cos x}(x + \operatorname{cosec} x) + c$$

Q.58

Sol. $I = \int \frac{5x^4 + 4x^5}{(x^5 + x + 1)^2} dx$

$$= \int \frac{5x^4 + 1}{(x^5 + x + 1)^2} dx + \int \frac{4x^5 - 1}{(x^5 + x + 1)^2} dx$$

$I = I_1 + I_2$

I_1

$\Rightarrow$ put $x^5 + x + 1 = t$

$$= \int \frac{1}{t^2} dt$$

$$= -\frac{1}{t} + c$$

$$= -\frac{1}{x^5 + x + 1} + c$$

I_2 (dividing by x^2)

$$\Rightarrow \int \frac{4x^3 - \frac{1}{x^2}}{\left(x^4 + 1 + \frac{1}{x}\right)^2} dx$$

put $x^4 + \frac{1}{x} + 1 = t$

$$\left(4x^3 - \frac{1}{x^2}\right) dx = dt$$

$$= \int \frac{1}{t^2} dt$$

$$= -\frac{1}{x^4 + \frac{1}{x} + 1} + c$$

$$= -\frac{-x}{x^5 + x + 1} + c$$

or $I = I_1 + I_2$

or $\boxed{I = -\frac{(x+1)}{x^5+x+1} + c}$

Q.60

Sol. $I = \int \frac{\cos^2 x}{1+\tan x} dx = \frac{1}{2}\int \frac{2\cos^3 x}{\sin x + \cos x} dx$

$$= \frac{1}{2}\int \frac{\cos^3 x - \sin^3 x + \cos^3 x + \sin^3 x}{(\sin x + \cos x)} dx$$

$$= \frac{1}{2}\int \frac{(\cos x - \sin x)\ (1+\sin x \cos x)}{\sin x + \cos x} dx + \frac{1}{2}\int \frac{(\cos x + \sin x)}{(\cos x + \sin x)}(1 - \sin x \cos x) dx$$

= put sinx + cosx = t

(cos x – sin x)dx = dt

$$= \frac{1}{2}\int \frac{\left[1 + \frac{1}{2}(t^2 - 1)\right]}{t} dt + \frac{1}{2}\int \left(1 - \frac{1}{2}\sin 2x\right) dx$$

$$= \frac{1}{2}[\log t] + \frac{1}{4}\left[\frac{t^2}{2} - \log t\right] + \frac{1}{2}x - \frac{1}{4}\frac{(-\cos 2x)}{2} + c$$

$$= \frac{(\sin x + \cos x)^2}{8} + \frac{1}{4}\log(\sin x + \cos x) + \frac{x}{2} + \frac{1}{8}\cos 2x + c$$

$$= \frac{1}{8} + \frac{\sin 2x}{8} + \frac{1}{4}\log(\sin x + \cos x) + \frac{x}{2} + \frac{\cos 2x}{8} + c$$

or $\boxed{I = \frac{(\sin 2x + \cos 2x)}{8} + \frac{1}{4}\log(\sin x + \cos x) + \frac{x}{2} + c}$

Q.61

Sol. $I = \int \frac{x^3 + x + 1}{x^4 + x^2 + 1} dx$

$$=\int \frac{x^3+x+1}{x^4+x^2+1+x^2-x^2}dx$$

$$=\int \frac{x^3+x+1}{(x^2+1)^2-x^2}dx$$

or $I=\int \frac{x^3+x+1}{(x^2+x+1)\,(x^2-x+1)}dx$

Now $\frac{x^3+x+1}{(x^2+1-x)\,(x^2+1+x)}=\frac{Ax+B}{(x^2+1-x)}+\frac{Cx+D}{x^2+1+x}$

on comparing, A = 0, B = $\frac{1}{2}$, C = 1, $D=\frac{1}{2}$

$$=\frac{x+\frac{1}{2}}{x^2+1+x}+\frac{\frac{1}{2}}{x^2+1-x}$$

or I = $\frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx+\frac{1}{2}\int \frac{1}{x^2-x+1+\frac{1}{4}-\frac{1}{4}}dx$

$$=\frac{1}{2}\log(x^2+x+1)+\frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

or $\boxed{I=\frac{1}{2}\log(x^2+x+1)+\frac{1}{2}.\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c}$

Q.62

Sol. $I=\int (\sin x)^{-11/3}(\cos x)^{-1/3}dx$

$$= \int \frac{(\sin x)^{1/3}}{\sin^4 x (\cos x)^{1/3}} dx$$

$$= \int \frac{\operatorname{cosec}^4 x}{(\cot x)^{1/3}} dx$$

$$= \int \frac{(1+\cot^2 x)\operatorname{cosec}^2 x}{(\cot x)^{1/3}} dx$$

$$= - \int \frac{1+t^2}{t^{1/3}} dt$$ put cot x = t $\Rightarrow$ $\operatorname{cosec}^2 x\, dx = - dt$

$$= - \left[\frac{t^{-1/3+1}}{\frac{-1}{3}+1} + \frac{t^{2-\frac{1}{3}+1}}{2-\frac{1}{3}+1} \right] + c$$

$$= - \left[\frac{3}{2} t^{2/3} + \frac{3}{8} t^{8/3} \right] + c$$

or $\boxed{I = -\left[\frac{3}{2}(\cot x)^{2/3} + \frac{3}{8}(\cot x)^{8/3} \right] + c}$

Q.63

Sol. $I = \int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx$

$$= \int \frac{1}{\sqrt{\sin^3 x [\sin x \cos\alpha + \cos x \sin\alpha]}} dx$$

$$= \int \frac{1}{\sqrt{\sin^4 x [\cos\alpha + \cot x \sin\alpha]}} dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot x . \sin\alpha + \cos\alpha}} dx$$

put $\sin\alpha . \cot x + \cos\alpha = t^2 \Rightarrow - \sin\alpha \operatorname{cosec}^2 x\, dx = 2t\, dt$

or $I = \int \frac{-1}{\sin\alpha}\frac{2t}{t}\,dt$

$= -\frac{2}{\sin\alpha}.\int 1.dt$

$= -\frac{2}{\sin\alpha}t + c$

$$\boxed{I = -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + c}$$

Q.64

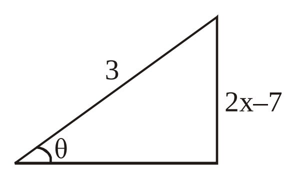
Sol. $I = \int \frac{x}{(7x-10-x^2)^{3/2}}\,dx$

$= \int \frac{x}{\left(\sqrt{\frac{1}{4}[9-(2x-7)^2]}\right)^3}\,dx$

put $2x - 7 = 3\sin\theta \Rightarrow 2dx = 3\cos\theta\, d\theta$

$= \frac{3}{4}\int \frac{3\sin\theta + 7}{\left(\sqrt{\frac{1}{4}(9-9\sin^2\theta)}\right)^3}\,d\theta$

$\sin\theta = \frac{2x-7}{3}$



$= \frac{3}{4}\int \frac{3\sin\theta + 7}{\left(\frac{3}{2}\cos\theta\right)^3}\,d\theta$

$\cos\theta = \frac{\sqrt{9-(2x-7)^2}}{3}$

$= \frac{3}{4}\times\frac{8}{27}\int \frac{3\sin\theta + 7}{\cos^3\theta}\,d\theta$

$\tan\theta = \frac{2x-7}{\sqrt{9-(2x-7)^2}}$

$= \frac{2}{9}\int \frac{3\sin\theta}{\cos^3\theta}\,d\theta + \frac{2}{9}\int \frac{7}{\cos^3\theta}\,d\theta$

$$= \frac{2}{3}\int \frac{\sin\theta}{\cos^3\theta}d\theta + \frac{14}{9}\int \sec^3\theta\, d\theta$$

put $\cos\theta = t \Rightarrow -\sin\theta\, d\theta = dt$ (using by part method)

$$= -\frac{2}{3}\int t^{-3}dt + \frac{14}{9}\sec\theta(\tan\theta - 1)$$

$$= \frac{1}{3}.\frac{1}{t^2} + \frac{14}{9}.\frac{3}{\sqrt{9-(2x-7)^2}}\left(\frac{2x-7}{\sqrt{9-(2x-7)^2}} - 1\right)$$

Q.65

Sol. $$I = \int \frac{dx}{\sec x + \operatorname{cosec} x}$$

$$= \int \frac{1}{2} \times \frac{2\sin x\cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x)^2 - 1}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\int (\sin x + \cos x)dx - \frac{1}{2}\int \frac{1}{\sin x + \cos x}dx$$

$$= \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2}\int \frac{1}{\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]}\, dx$$

$$= \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2\sqrt{2}}\int \frac{1}{\sin\left(x + \frac{\pi}{4}\right)}dx$$

$$= \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2\sqrt{2}}\int \operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx$$

$$\boxed{I = \frac{1}{2}(-\cos x + \sin x) - \frac{1}{2\sqrt{2}}\log\tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c}$$

Q.66

Sol. $I = \int \frac{dx}{\sin x + \sec x} dx$

$$= \int \frac{\cos x}{1 + \sin x \cos x} dx$$

$$= \int \frac{2\cos x}{2 + 2\sin x \cos x} dx$$

$$= \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + \sin 2x} dx$$

$$= \int \frac{\cos x + \sin x}{2 + \sin 2x} dx + \int \frac{\cos x - \sin x}{2 + \sin 2x} dx$$

$$= \int \frac{\cos x + \sin x}{2 + \{1 - (\sin x - \cos x)^2\}} dx + \int \frac{\cos x - \sin x}{2 + \{(\sin x + \cos x)^2 - 1\}} dx$$

put $\sin x - \cos x = u$ put $\sin x + \cos x = v$

$(\cos x + \sin x)dx = du$ $(\cos x - \sin x)dx = du$

$$= \int \frac{1}{(\sqrt{3})^2 - u^2} du + \int \frac{dv}{1^2 + v^2}$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{\sqrt{3} + u}{\sqrt{3} - u}\right| + \tan^{-1} v + c$$

$$\boxed{I = \frac{1}{2\sqrt{3}} \log\left|\frac{\sqrt{3} + \sin x - \cos x}{\sqrt{3} - \sin x + \cos x}\right| + \tan^{-1}(\sin x + \cos x) + c}$$

Q.67

Sol. $I = \int \frac{x^2 + 1}{x^4 - 2x^2 \cos\alpha + 1} dx$

divide by x^2 on N^r and D^r

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2} - 2\cos\alpha + 2 - 2} dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2 + 2(1-\cos\alpha)} dx$$

$$= \int \frac{dt}{t^2 + \left(2\sin\frac{\alpha}{2}\right)^2}$$

put $x - \frac{1}{x} = t$

$\left(1+\frac{1}{x^2}\right)dx = dt$

or $\boxed{I = \frac{1}{2}\left(\operatorname{cosec}\frac{\alpha}{2}\right)\tan^{-1}\left(\frac{x-\frac{1}{x}}{2\sin\frac{\alpha}{2}}\right) + c}$

Q.68

Sol. $I = \int \frac{\cos x - \sin x}{7 - 9\sin 2x} dx$

or $I = \int \frac{dt}{7 - 9(t^2 - 1)}$

Let $\boxed{\sin x + \cos x = t} \Rightarrow (\cos x - \sin x)dx = dt$

$= \int \frac{dt}{4^2 - (3t)^2}$

$\because \sin^2 x + \cos^2 x + 2 \sin x \cos x = t^2$

or $\sin^2 x + \cos^2 x + 2\sin x \cos x = t^2$

$= \int \frac{dt}{4^2 - (3t)^2}$

or $1 + \sin 2x = t^2$

or $\boxed{\sin 2x = t^2 - 1}$

$= \frac{1}{2.4} \cdot \frac{1}{3} \ell n \left| \frac{4+3t}{4-3t} \right| + c$

or $\boxed{I = \frac{1}{24}\ell n\left|\frac{4+3(\sin x+\cos x)}{4-3(\sin x+\cos x)}\right| + c}$

Q.69

Sol. $I = \int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}\, dx$

$= \int \frac{(\cos x - \sin x)}{\sqrt{\sin x \cos x}(1+3\sin 2x)}\, dx$

$= \int \frac{1}{\sqrt{\frac{t^2-1}{2}}(3t^2-2)} dt$ put sin x + cos x = t

(cos x – sin x) dx = dt

$= \sqrt{2}\int \frac{1}{(3t^2-2)\sqrt{t^2-1}} dt$ & 1 + sin 2x = $t^2 \Rightarrow$ sin 2x = t^2–1

put $t = \frac{1}{u} \Rightarrow dt = -\frac{1}{u^2} du$

$\sqrt{2}\int \frac{-du}{u^2\left(\frac{3}{u^2}-2\right)\sqrt{\frac{1}{u^2}-1}} = -\sqrt{2}\int \frac{u\,du}{(3-2u^2)\sqrt{1-u^2}}$

Q.70

Sol. $I = \int \frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^2+1)^2}\, dx$

$= \int \frac{4x^5-7x^4+8x^3-2x^2+4x-7}{x^2(x^4+2x^2+1)}\, dx$

$= \int \frac{4x(x^4+2x^2+1)-7(x^4+1+2x^2)+12x^2}{x^2(x^4+2x^2+1)}\, dx$

$= \int \frac{4}{x}\, dx - \int \frac{7}{x^2} dx + \int \frac{12}{(x^2+1)^2}\, dx$ put x = tanθ $\Rightarrow$ dx = $\sec^2\theta d\theta$

$$= 4\log x - 7\left(-\frac{1}{x}\right) + \int \frac{12\sec^2\theta d\theta}{(1+\tan^2\theta)^2}$$

$$= 4\log x + \frac{7}{x} + 12\int \frac{1}{\sec^2\theta} d\theta$$

$$= 4\ \ell n|\ x\ | + \frac{7}{x} + 12\int \cos^2\theta\, d\theta$$

$$= 4\ \ell n|\ x\ | + \frac{7}{x} + 12\int \left(\frac{\cos 2\theta + 1}{2}\right) d\theta$$

$$= 4\ell n\ |\ x\ | + \frac{7}{x} + 6\theta + 6\ \frac{\sin^2\theta}{2} + c$$

$$= 4\ell n|x| + \frac{7}{x} + 6\tan^{-1}x + 3\sin(2\tan^{-1}x) + c$$

$$= 4\ell n|\ x\ | + \frac{7}{x} + 6\tan^{-1}x + 3\left[\frac{2\tan(\tan^{-1}x)}{1+[\tan(\tan^{-1}x)]^2}\right] + c$$

$$\boxed{I = 4\ell n\ |\ x\ | + \frac{7}{x} + 6\tan^{-1}x + \frac{6x}{1+x^2} + c}$$

Q.71

Sol. $I = \int \sqrt{\frac{(1-\sin x)\ (2-\sin x)}{(1+\sin x)\ (2+\sin x)}} dx$

$$= \int \frac{\cos x}{(1+\sin x)} \frac{\sqrt{4-\sin^2 x}}{2+\sin x} dx$$

$$= \int \frac{\sqrt{4-(t-1)^2}}{t(1+t)} dt$$ put $1 + \sin x = t \Rightarrow \cos x\, dx = dt$

$$= \int \frac{\sqrt{3+2t-t^2}}{t(1+t)} dt = \int \frac{-(t-3)\ (t+1)}{t(1+1)\sqrt{3+2t-t^2}} dt$$

$$\text{or } I = \int \frac{(3-t)}{t\sqrt{3+2t-t^2}}\,dt$$

$$\text{or } I = 3\int \frac{1}{t\sqrt{3+2t-t^2}} - \int \frac{1}{\sqrt{3+2t-t^2}}\,dt$$

$$\text{put } t = \frac{1}{v}$$

$$dt = -\frac{1}{v^2}\,dv$$

$$\text{or } I = 3\int \frac{-1}{v^2}.\frac{dv}{\frac{1}{v}\sqrt{3+\frac{2}{v}-\frac{1}{v^2}}} - \int \frac{1}{\sqrt{(2)^2-(t-1)^2}}\,dt$$

$$= -3\int \frac{dv}{\sqrt{3v^2+2v-1}} - \int \frac{1}{\sqrt{(2)^2-(t-1)^2}}\,dt$$

$$= -3\int \frac{dv}{\sqrt{\left(\sqrt{3}v-\frac{1}{\sqrt{3}}\right)^2-\left(\frac{2}{\sqrt{3}}\right)^2}} - \int \frac{1}{\sqrt{(2)^2-(t-1)^2}}\,dt$$

$$= -3\sqrt{3}\int \frac{dv}{\sqrt{(3v-1)^2-(2)^2}} - \int \frac{1}{\sqrt{(2)^2-(t-1)^2}}\,dt$$

$$= -\frac{3\sqrt{3}}{3}\log\left[(3v-1)+\sqrt{(3v-1)^2-4}\right] - \sin^{-1}\left(\frac{t-1}{2}\right)+c$$

$$= -\sqrt{3}\log\left[\left(\frac{3}{1+\sin x}-1\right)+\sqrt{\left(\frac{3}{1+\sin x}-1\right)^2-4}\right] - \sin^{-1}\left(\frac{\sin x}{2}\right)+c$$

Q.72

Sol. $I = \int \frac{dx}{\cos^3 x - \sin^3 x}$

$$= \int \frac{1}{(\cos x - \sin x)\left(1 + \frac{\sin 2x}{2}\right)} dx$$

$$= \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)^2 \left(1 + \frac{\sin 2x}{2}\right)} dx$$

$$= \int \frac{(\cos x - \sin x)}{(1 - \sin 2x)\left(1 + \frac{\sin 2x}{2}\right)} dx$$

$$= \int \frac{(\cos x - \sin x)}{(1 - \sin 2x)\left(1 + \frac{\sin 2x}{2}\right)} dx$$

$$= 2\int \frac{(\cos x - \sin x)}{[2 - (\sin x + \cos x)^2]\,[1 + (\sin x + \cos x)^2]} dx$$

put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)dx = dt$

$$= 2\int \frac{dt}{(2 - t^2)\,(1 + t^2)}$$

$$= \frac{2}{3}\int \left(\frac{1}{1 + t^2} + \frac{1}{2 - t^2}\right) dt$$

$$= \frac{2}{3}\int \frac{1}{1 + t^2} dt + \frac{2}{3}\int \frac{1}{2 - t^2} dt$$

$$= \frac{2}{3}\tan^{-1} t + \frac{2}{3}.\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2} + t}{\sqrt{2} - t}\right| + c$$

or $\boxed{I = \frac{2}{3}\tan^{-1}(\sin x + \cos x) + \frac{1}{3\sqrt{2}}\log\left|\frac{\sqrt{2} + (\sin x + \cos x)}{\sqrt{2} - (\sin x + \cos x)}\right| + c}$

Q.73

Sol. $I = \int \frac{dx}{(x - \alpha)\sqrt{(x - \alpha)(x - \beta)}}$

put $x - \alpha = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$

$$= \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t}\left[\frac{1}{t}+\alpha-\beta\right]}}$$

$$= -\int \frac{dt}{\sqrt{1+(\alpha-\beta)t}}$$

$$= -\int \frac{1}{u}.\frac{2u\,du}{(\alpha-\beta)}$$

put $1+(\alpha-\beta)\,t = u^2 \Rightarrow (\alpha-\beta)dt = 2udu \Rightarrow dt = \frac{2u}{(\alpha-\beta)}du$

$$= -\frac{2}{(\alpha-\beta)}.u + c$$

$$= -\frac{2}{(\alpha-\beta)}\sqrt{1+(\alpha-\beta)t} + c$$

or $\boxed{I = -\frac{2}{(\alpha-\beta)}\sqrt{1+\frac{(\alpha-\beta)}{(x-\alpha)}} + c}$

or $\boxed{I = -\frac{2}{(\alpha-\beta)}\sqrt{\frac{(x-\beta)}{(x-\alpha)}} + c}$

Q.74

Sol. $I = \int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$

$$= \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}}\,dx$$

$$= \int \sqrt{\cot^2 x - 1}\,dx$$

putting $\cot x = \sec\theta \Rightarrow -\text{cosec}^2 x\,dx = \sec\theta\tan\theta\,d\theta$

or $I = \int \sqrt{\sec^2\theta - 1} \times \frac{\sec\theta\tan\theta}{-\text{cosec}^2 x} d\theta$

$\cot x = \sec\theta \Rightarrow 1 + \cot^2 x = 1 + \sec^2\theta \Rightarrow \text{cosec}^2 x = 1 + \sec^2\theta$

$= -\int \frac{\sec\theta\tan^2\theta}{1+\sec^2\theta} d\theta$

$= \int \frac{\sec^2\theta}{\cos\theta + \cos^3\theta} d\theta$

$= -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)} d\theta$

$= -\int \sec\theta\, d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta} d\theta$

$= -\int \sec\theta\, d\theta + 2\int \frac{\cos\theta}{2-\sin^2\theta} d\theta$

put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$= -\log|\sec\theta + \tan\theta| + 2 \times \frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + c$

$= -\log|\sec\theta + \tan\theta| + \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+\sqrt{1-\cos^2\theta}}{\sqrt{2}-\sqrt{1-\cos^2\theta}}\right| + c$

or $\boxed{I = -\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^x}}\right| + c}$

Q.75

Sol. $I = \int \frac{\sqrt{\sin^4 x + \cos^4 x}}{\sin^3 x \cos x} dx$

$= \int \frac{\sin^2 x\sqrt{1+\cot^4 x}}{\sin x.\sin^3 x.\frac{\cos x}{\sin x}} dx$

$$= \int \frac{\sqrt{1+(\cot^2 x)^2}}{\sin^2 x.\cot x}dx$$

put $\cot^2 x = t \Rightarrow 2\cot x. (-\operatorname{cosec}^2 x)\, dx = dt$

$$= \int \frac{-\sqrt{1+t^2}}{2\cot x.\cot x}dt$$

$$= -\frac{1}{2}\int \frac{\sqrt{1+t^2}}{t}\, dt$$

$$= -\frac{1}{2}\int \frac{(1+t^2)}{t\sqrt{1+t^2}}dt$$

$$= -\frac{1}{2}\left[\int \frac{1}{t\sqrt{1+t^2}}dt + \int \frac{t}{\sqrt{1+t^2}}dt\right]$$

put $t = \frac{1}{u}$ put $1 + t^2 = v^2$

$2t\, dt = 2v\, dv$

$$= -\frac{1}{2}\left[\int \frac{u^2}{\sqrt{1+u^2}}du + \int \frac{v\,dv}{v}\right]$$

$$= -\frac{1}{2}\left[\int \sqrt{1+u^2}dx - \int \frac{1}{\sqrt{1+u^2}}du + v\right]$$

$$= -\frac{1}{2}\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ell n(u+\sqrt{u^2+1}) - \ell n(u+\sqrt{u^2+1}) + \sqrt{1+t^2}\right]$$

$$= \left(-\frac{u}{4}\sqrt{1+u^2} + \frac{1}{4}\ell n(4+\sqrt{u^2+1}) - \frac{\sqrt{1+t^2}}{2}\right)$$

$$= -\frac{\tan^2 x}{4}.\sec x + \frac{1}{u}\ell n(\tan^2 x + \sqrt{1+\tan^4 x}) - \frac{1}{2}\sec x + c$$

Q.76

Sol. $I = \int \frac{1-(\cot x)^{2008}}{\tan x + (\cot x)^{2009}} dx = \frac{1}{k} \ell n |\sin^k x + \cos^k x| + C$

$$\Rightarrow \int \frac{1-\left(\frac{\cos x}{\sin x}\right)^{2008}}{\frac{\sin x}{\cos x} + \left(\frac{\cos x}{\sin x}\right)^{2009}} dx$$

$$\Rightarrow \int \frac{\sin^{2008} x - \cos^{2008} x}{\sin^{2008} x \frac{(\sin^{2010} x - \cos^{2010} x)}{\sin^{2009} x \cos x}} dx$$

$$= \int \frac{(\sin^{2008} x - \cos^{2008} x)\sin x \cos x}{\sin^{2010} x + \cos^{2010} x} dx$$

put $\sin^{2010} x + \cos^{2010} x = t \Rightarrow [(2010)\sin^{2009} x. \cos x + 2010 \cos^{2009} x (-\sin x)]\, dx = dt$

$\Rightarrow (2010) \sin x. \cos x [\sin^{2008} x - \cos^{2008} x]dx = dt$

$$= \frac{1}{2010}\int \frac{1}{t} dt$$

$$= \frac{1}{2010} \log|t| + c$$

or $\Rightarrow \frac{1}{2010} \log|\sin^{2010} x + \cos^{2010} x| + c = \frac{1}{k} \log|\sin^{2010} x + \cos^{2010} x| + c$

$\boxed{k = 2010}$ **Ans.**

Q.77

Sol. $I = \int \cos 2\theta . \ell n \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} d\theta$

$$= \frac{1}{2}\int \cos 2\theta \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)^2 d\theta$$

$$= \frac{1}{2}\int \cos 2\theta \log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right) d\theta$$

$$= \frac{1}{4}\int \log\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right).2\cos 2\theta\, d\theta$$

put $\sin 2\theta = t$

$2\cos 2\theta \, d\theta = dt$

$= \frac{1}{4}\int \log\left(\frac{1+t}{1-t}\right) dt$

$= \frac{1}{4}\left[\int \log(1+t)dt - \int \log(1-t)dt\right]$

$= \frac{1}{4}$ [t log (1 + t) – t + log (1 + t) – t log (1 – t) + t + log (1 – t)]

$= \frac{1}{4}\left[t\log\left(\frac{1+t}{1-t}\right) + \log(1-t^2)\right] + c$

$= \frac{1}{4}\left[\sin 2\theta \log\left(\frac{1+\sin 2\theta}{1-\sin\theta}\right) - \frac{1}{2}\ell n(\sec^2 2\theta) + c\right]$

or $\boxed{I = \frac{1}{2}(\sin 2\theta)\log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \frac{1}{2}\ell n(\sec 2\theta) + c}$

Q.78

Sol. $I = \int \frac{x^2}{(x\cos x - \sin x)\,(x\sin x + \cos x)}\, dx$

$I = \int \left[\frac{x\cos x}{(x\sin x + \cos x)} + \frac{x\sin x}{x\cos x - \sin x}\right] dx$

$= \int \frac{x\cos x}{x\sin x + \cos x}\, dx \quad + \quad \int \frac{x\sin x}{x\cos x - \sin x} dx$

I_1 $\qquad\qquad\qquad$ I_2

put x sin x + cos x = t $\qquad$ put x cosx – sin x = t

(x cos x + sin x – sin x) dx = dt $\qquad$ (–x sin x + cos x – cos x) dx = dt

x cos x dx = dt $\qquad$ – x sin x dx = dt or $I_1 = \int \frac{1}{t} dt$ $\qquad$ $I_2 = -\int \frac{1}{t} dt$

$= \ell n\,|x \sin x + \cos x| + c$ $\qquad$ $= -\ell n|x \cos x - \sin x| + c$

or $I = I_1 + I_2$

$$\boxed{I = \ell n\left|\frac{x\sin x + \cos x}{x\cos x - \sin x}\right| + c}$$ **Ans**