FUNCTIONS

EXERCISE - 1(C)

Q.1 [03]

$\sin\frac{2x}{3}+\cos 4x+|\tan 3x|+\text{sgn}(x^2+4x+15)$ has period as LCM of $\left(\frac{2\pi\times 3}{2}\ ,\ \frac{2\pi}{4}\ ,\ \frac{\pi}{3}\right)$

$\because \text{sgn}(x^2+4x+15)=1$ as $x^2+4x+15>0$ for all $x,$ so period can be any real number.

LCM of $\left(3\pi, \frac{2\pi}{2}, \frac{\pi}{3}\right)$ is 3π.

So, k = 3.

Q.2 [05]

$[x]-\{x\}=\frac{x}{3} \Rightarrow 3([x]-\{x\})=[x]+\{x\}$

$\Rightarrow \{x\} = \frac{[x]}{2}$

$\because 0 \le \{x\} < 1 \Rightarrow 0 \le \frac{[x]}{2} < 1$

$\Rightarrow [x]=0,1$ & $\{x\}=0, \frac{1}{2}$

So, $x=\{x\}+[x]$ gives $x=0, \frac{3}{2}$

So, sum of values of x, $\lambda=0+\frac{3}{2}$

Hence, value of $\frac{10\lambda}{3}=\frac{10}{3}\times\frac{3}{2}=5$

Q.3 [02]

$f(x)+f(y)+f(xy)=2+f(x)\cdot f(y)$

at x = 1 , y = 1, $3f(1)=2+f(1)^2$

$\Rightarrow f(1)^2 - 3f(1) + 2 = 0$

$\Rightarrow f(1) = 2$ or $f(1) = 1$.

Now at y = 1, $f(x) + f(1) + f(x) = 2 + f(x) \cdot f(1)$

$\Rightarrow f(x)(2 - f(1)) = 2 - f(1)$

$$\Rightarrow f(x) = \frac{2 - f(1)}{2 - f(1)}$$

Hence if *f* (1) = 1, then *f* (x) = 1.

If f (x) = 2, then substitute, y = 1/x to get $f(x) + f\left(\frac{1}{x}\right) + f(1) = 2 + f(x) \cdot f\left(\frac{1}{x}\right)$

$$\Rightarrow f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$$

$$\Rightarrow f(x) = 1 + \frac{1}{f\left(\frac{1}{x}\right) - 1}$$

Solution of such polynomial is, $f(x) = 1 \pm x^n$ but, $f(1) = 2 \Rightarrow f(x) = 1 + x^4$

but $f(4) = 17 \Rightarrow 1 + 4^n = 17 \Rightarrow n = 2$

$$f(5) = \frac{5^2 + 1}{13} = \frac{26}{13} = 2.$$

Q.4 [01]

$$\left(\frac{x}{1+x^2}\right)^2 + a\left(\frac{x}{1+x^2}\right) + 3 = 0 \Rightarrow \frac{1}{\left(x + \frac{1}{x}\right)^2} + \frac{a}{\left(x + \frac{1}{x}\right)} + 3 = 0$$

$$\Rightarrow 3\left(x + \frac{1}{x}\right)^2 + a\left(x + \frac{1}{x}\right) + 1 = 0.$$

Let $x + \frac{1}{x} = t$, then $\Rightarrow 3t^2 + at + 1 = 0$.

Now range of $x+\frac{1}{x}$ is $(-\infty,-2]\cup[2,\infty)$

Every root of $f(t)=3t^2+at+1=0$ which lies in $(-\infty,-2)\cup(2,\infty)$ gives two values of x and t = 2 or –2 gives one value of x.

Hence exactly two distinct roots are possible when exactly one root lies in (-2, 2) and other root is not equal to -2 or 2.

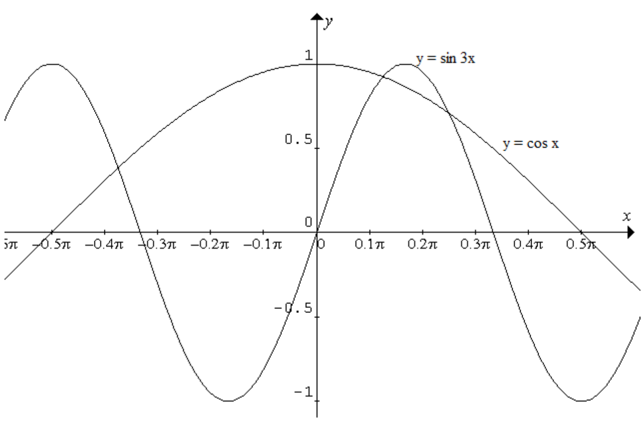
Thus $f(-2)\,f(2)<0$ & $f(\pm 2)\neq 0$

$\Rightarrow (13-2a)(13+2a)<0$

$\Rightarrow a<-\frac{13}{2}$ or $a>\frac{13}{2}$

Hence $\lambda=\mu=\frac{13}{2}\Rightarrow\frac{\lambda+\mu}{13}=1$.

Q.5 [03]



Refer the adjoining graph of

$y=\cos x$ & $y=\sin 3x$

Number of points intersection in

$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

k = 3

Q.6 [05]

$f(x)=\sqrt{8x-x^2}-\sqrt{14x-x^2-48}$

$=\sqrt{(8-x)x}-\sqrt{(8-x)(x-6)}$

Domain : $6\leq x\leq 8$

Now $f(x)=\sqrt{8-x}\left(\sqrt{x}-\sqrt{x-6}\right)$

$$\Rightarrow f'(x) = -\frac{\sqrt{x} - \sqrt{x-6}}{2\sqrt{8-x}} + \sqrt{8-x}\left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x-6}}\right)$$

$$\Rightarrow f'(x) = \left(\sqrt{x-6} - \sqrt{x}\right)\left(\frac{\sqrt{x-6}\sqrt{x} + 8 - x}{2\sqrt{8-x}\sqrt{x-6}\sqrt{x}}\right)$$

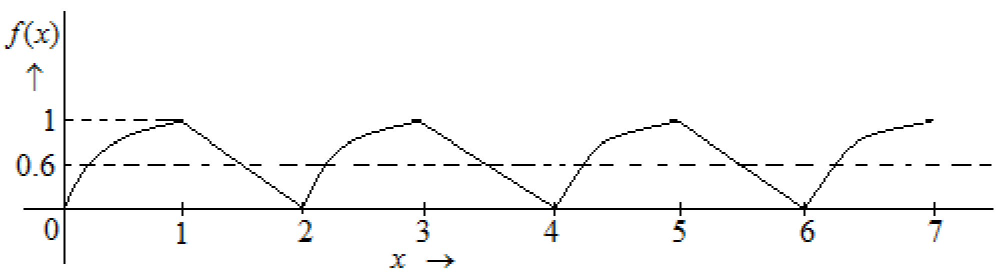
Now $\sqrt{x-6} < \sqrt{x}$ & $\sqrt{x-6}\sqrt{x} > (x-8) \Rightarrow f'(x) < 0$ for $6 \le x \le 8$

Hence $f_{MAX} = f(6) = \sqrt{12}$ & $f_{MIN} = f(8) = 0$.

Thus $m\sqrt{n} = 2\sqrt{3}$.

Q.7 [02]

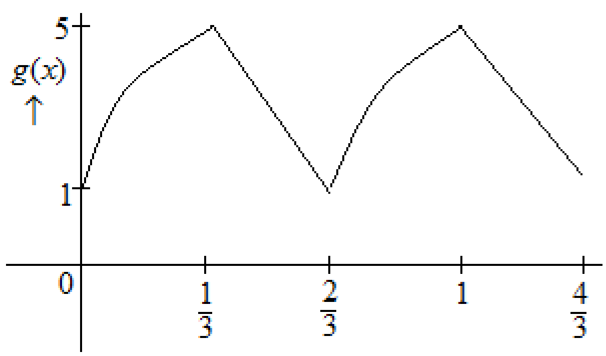
Given $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 2 - x & 1 \le x < 2 \\ f(x+2) & \text{for all } x \end{cases}$



f (x) = 0.6 : $\sqrt{x} = 0.6 \quad \Rightarrow x = (0.6)^2 = 0.36$, so sum $= 4 + 6 + 2 \times 0.36 = 10.72$

& $2 - x = 0.6 \Rightarrow x = 0.4$, so sum $= 3 + 0.4 + 5 + 0.4 = 8.8$

A $= 10.72 + 8.8 = 19.52$



Now $g(x) = 4f(3x) + 1 \ \forall \ x \in R$

$$\Rightarrow g(x) = \begin{cases} 4\sqrt{3x} + 1 & x \in \left[0, \frac{1}{3}\right) \\ 3 - 4x & x \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ f(3x+2) & x \in \text{all} \end{cases}$$

Fundamental Period $=\left(\frac{2}{3}\right) \Rightarrow$ B $=\frac{2}{3}$.

$g(x)=4f(3x)+1 \Rightarrow g'(x)=12f'(2x)+0$

or $g'\left(\frac{13}{2}\right)=12x\ f'\left(\frac{39}{2}\right)$

g;(6.5) = – 12

So, $|C|=12$

Hence, $\frac{[A]\ B\,|C|}{76}=17\times\frac{2}{3}\times\frac{12}{76}=2$

Q.8 [05]

$x^4-4x^3+6x^2-4x=2008 \Rightarrow (x-1)^4=2009$

$\Rightarrow (x-1)=(2009)^{\frac{1}{4}},-(2009)^{\frac{1}{4}},\ (2009)^{\frac{1}{4}}i\ ,\ -(2009)^{\frac{1}{4}}i$

So, non-real roots $=1\pm(2009)^{\frac{1}{4}}\cdot i$

product of non-real roots, P $=\left[1+(2009)^{\frac{1}{4}}\cdot i\right]\left[1-(2009)^{\frac{1}{4}}i\right]$

$P=1+(2009)^{\frac{1}{2}}$

So, $[P]=\left[1+(2009)^{\frac{1}{2}}\right]=45$.

Q.9 [03]

Given $f\left(\frac{2x-3}{x-2}\right)=5x-2\ ,\ x\neq 2$

$\Rightarrow$ let , $\frac{2x-3}{x-2}=t$

$\Rightarrow 2x-3=tx-2t$ or $x=\frac{2t-3}{t-2}$

$$\Rightarrow f(t) = 5\left(\frac{2t-3}{t-2}\right) - 2$$

$$\Rightarrow f(t) = \frac{8t-17}{t-2}$$

So, $f(x) = \frac{8x-11}{x-2}$

Now let $y = \frac{8x-11}{x-2}$

$$\Rightarrow x = \left(\frac{2y-11}{y-8}\right)$$

So, $f^{-1}(x) = \frac{2x-11}{x-8}$

$$f^{-1'}(13) = \frac{26-11}{5} = \frac{15}{5} = 3$$

Q.10 [04]

$\because P(x)$ has odd degree terms only so $P(-x) = -P(x)$

$P(x)$ divided by $(x-3)$ gives remainder 6 hence $P(3) = 6$

$P(x)$ divided by $(x+3)$ will give remainder $P(-3) = -P(3) = -6$

Now let $P(x) = \left(x^2 - 9\right)Q\left(x\right) + Ax + B$, where g(x) = Ax + B

So, $P(3) = 6 \quad \Rightarrow 3A + B = 6$

& $P(-3) = -6 \Rightarrow -3A + B = -6$

Solving simultaneously gives A = 2, B = 0.

g(2) = 4.

Q.11 [04]

$$f : R \to \left(0, \frac{2\pi}{3}\right], \; f(x) = \cot^{-1}(x^2 - 4x + \alpha)$$

For f(x) to be an ONTO function, $0 \le \cot^{-1}(x^2 - 4x + \alpha) \le \frac{2\pi}{3}$ for all real x.

or $x^2 - 4x + \alpha \ge \cot\left(\frac{2\pi}{3}\right)$.

$\Rightarrow x^2 - 4x + \alpha \ge -\frac{1}{\sqrt{3}}$.

$\Rightarrow x^2 - 4x + \left(\alpha + \frac{-1}{\sqrt{3}}\right) \ge 0$ for all real x.

So, $D \le 0 \Rightarrow 16 - 4\left(\alpha + \frac{1}{\sqrt{3}}\right) \le 0$.

$\Rightarrow \alpha \ge 4 - \frac{4}{\sqrt{3}}$.

So, smallest integral value of α is 4.

Q.12 [04]

$f(x) = \sin^{-1}x + \tan^{-1}x + x^2 + 4x + 1 \Rightarrow f(x) = \sin^{-1}x + \tan^{-1}x + (x+2)^2 - 3$

Now for $x \in [-1, 1]$, all of $\sin^{-1}x, \tan^{-1}x$ & $(x+2)^2$ are increasing functions.

Hence p = f(-1) & q = f(1)

Therefore p + q = 4.

Q.13 [00]

$\log_{\sin x} 2^{\tan x} > 0$

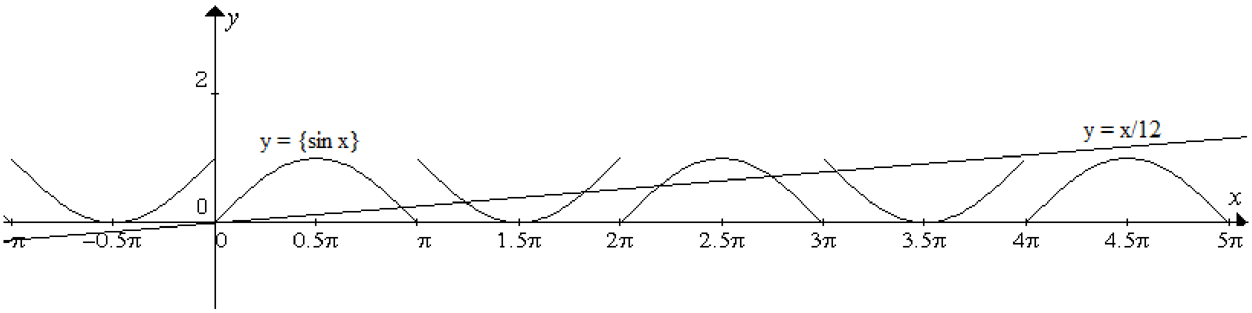
$\Rightarrow (\tan x) \cdot \log_{\sin x} 2 > 0$

$\Rightarrow \frac{\tan x}{(\log_2 \sin x)} > 0$

$\tan x > 0$ & $\log_2(\sin x) < 0$ in $\left(0, \frac{\pi}{2}\right)$ hence no solution.

{ $\log_a b$ is negative if a > 0 & 0 < a < 1}

Q.14 [07]



$12\{\sin x\} - x = 0$

$\Rightarrow \{\sin x\} = \left(\dfrac{x}{12}\right)$

Refer the adjoining graph.

Q.15 [04]

$[x]+2\{-x\}=3x \Rightarrow [x]+2\{-x\}=3[x]+3\{x\}$

Case I : For, $x\in$ I, $\{-x\} = \{x\}=0$

$\Rightarrow [x]=3[x]$

$\Rightarrow [x] = 0$

$\Rightarrow x=0$

Case II : For $x\notin$ I, $[x]+2(1-\{x\}) = 3[x]+3\{x\}$

$\Rightarrow \{x\} = \dfrac{2-2[x]}{5}$

Now $0 \le \{x\} < 1$, hence $0 \le \dfrac{2-2[x]}{5} < 1$

$\Rightarrow -2 \le -2[x] < 3$

$\Rightarrow -\dfrac{3}{2} < [x] \le -1$

So, $[x]=1, [x]=0, [x]=-1$

$\{x\}=0, \{x\}=\dfrac{2}{5}$, $\{x\}=\dfrac{4}{5}$

So, $x=1$, $x=\frac{2}{5}, x=-\frac{1}{5}$

Q.16 [02]

$(x)=[x]+1 \ : x \notin \mathrm{I}$

Hence, $[x]^2+([x]+1)^2<4$

$\Rightarrow 2[x]^2+2[x]-3<0$

So, $[x]\in\left(\frac{-1-\sqrt{7}}{2}\ ,\ \frac{-1+\sqrt{7}}{2}\right)$

So, $x\in[-1,1)$

Length of interval = 2

Q.17 [02]

$$g(x)=\left(4\cos^4 x-2\cos 2x-\frac{1}{2}\cos 4x-x^7\right)^{\frac{1}{7}}$$

$$\Rightarrow g(x)=\left[4\cos^4 x-4\cos^2 x+2-\frac{1}{2}(2\cos^2 2x-1)-7\right]^{\frac{1}{7}}$$

$$=\left[4\cos^4 x-4\cos^2 x+2-\cos^2 2x+\frac{1}{2}-x^7\right]^{\frac{1}{7}}$$

$$=\left[4\cos^4 x-4\cos^2 x-(2\cos^2 x-1)^2+\frac{3}{2}-x^7\right]^{\frac{1}{7}}$$

$$=\left[4\cos^4 x-4\cos^2 x-4\cos^4 x+4\cos^2 x-1+\frac{3}{2}-x^7\right]^{\frac{1}{7}}$$

$$=\left(\frac{1}{2}-x^7\right)^{\frac{1}{7}}$$

So, $g(g(x))=\left[\frac{1}{2}-\left(\frac{1}{2}-x^7\right)^{\frac{1}{7}\times 7}\right]^{\frac{1}{2}}$

$$=\left(\frac{1}{2}-\frac{1}{2}+x^7\right)^{\frac{1}{7}}$$

$$=x$$

So, $\frac{g(g(100))}{50}=\frac{100}{50}=2$

Q.18 [01]

$f(x)=\frac{3x-2}{x+4}=y \Rightarrow 3x-2=xy+4y$

$\Rightarrow x=\left[\frac{4y+2}{3-y}\right]$

So, $f^{-1}(x)=\frac{4x+2}{3-x}=\frac{x+\frac{1}{2}}{\frac{3}{4}-\frac{x}{4}}$.

Hence $b=\frac{1}{2}, c=-\frac{1}{4}\ \&\ d=\frac{3}{4} \Rightarrow b+c+d=1.$

Q.19 [02]

$f(x)=\frac{ax^8+bx^6+cx^4+dx^2+15x+1}{x}$

$f(x)=-f(x)$

Hence, $f(-5)=-f(5)=-(-28)=28$

So, $f\frac{(-5)}{14}=\frac{28}{14}=2$.

Q.20 [01]

$$\log_2(3-x)+\log_{\frac{1}{2}}\left(\frac{\sin\frac{9\pi}{4}}{5-x}\right) = \cos\frac{11\pi}{3} - \log_{\frac{1}{2}}(x+7)$$

Domain : $x < 3$, $x > -7$

Sol : $\log_2(3-x)+\log_{\frac{1}{2}}\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} - \log_{\frac{1}{2}}(5-x) = \frac{1}{2} - \log_{\frac{1}{2}}(x+7)$

$\Rightarrow \log_2(3-x)+\log_2(5-x)-\log_2(x+7)=0$

$\Rightarrow \frac{(3-x)(5-x)}{x+7}=1$

$\Rightarrow x^2-8x+15=x+7$

$\Rightarrow x^2-9x+8=0$

$\Rightarrow (x-1)\ (x-8)=0$

$\Rightarrow x=1$, $x=8$ but, $x\in(-7, 3)$, hence only one integral value of x is possible.