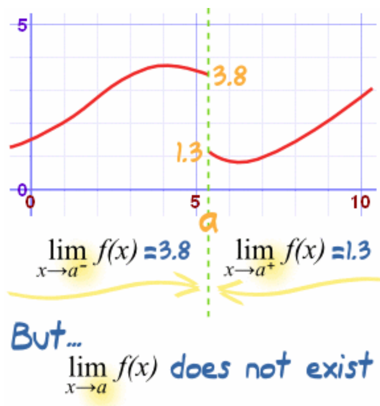
PREFACE

In mathematics, the **limit of a function** is a fundamental concept in calculus and analysis concerning the behaviour of that function near a particular input.

Informally, a function f assigns an output f(x) to every input x. We say the function has a limit L at an input p: this means f(x) gets closer and closer to L as x moves closer and closer to p. More specifically, when f is applied to any input sufficiently close to p, the output value is forced arbitrarily close to L. On the other hand, if some inputs very close to p are taken to outputs that stay a fixed distance apart, we say the limit does not exist.

The notion of a limit has many applications in modern calculus. In particular, the many definitions of continuity employ the limit: roughly, a function is continuous if all of its limits agree with the values of the function. It also appears in the definition of the derivative: in the calculus of one variable, this is the limiting value of the slope of secant lines to the graph of a function.



Although implicit in the development of calculus of the 17th and 18th centuries, the modern idea of the limit of a function goes back to Bolzano who, in 1817, introduced the basics of the epsilon-delta technique to define continuous functions. However, his work was not known during his lifetime (Felscher 2000). Cauchy discussed limits in his Course d'analyse (1821) and gave essentially the modern definition, but this is not often recognized because he only gave a verbal definition (Grabiner 1983). Weierstrass first introduced the epsilon-delta definition of limit in the form it is usually written today. He also introduced the notations **lim** and $\mathbf{lim}_{x\to x0}$ (Burton 1997).

The modern notation of placing the arrow below the limit symbol is due to Hardy in his book A Course of Pure Mathematics in 1908 (Miller 2004).

Have fun
Mathematics Department
IIT-ian's Pace

CONTENTS

LIMITS : Tentative Lecture Flow (Board Syllabus & Booklet Discussion Included)	
Lecture no.1	Limit definition, LHL & RHL, existence of limit, algebra of limit, different type of indeterminate forms
Lecture no.2	limit of trigonometric, rational, polynomial, logarithmic, expotential functions , sandwich theorem
Lecture no.3	Various methods of solving indeterminate forms of limits, sandwich theorem, L' hospital rule, expansion of functions

LIMIT OF A FUNCTION

1. NEIGHBOURHOOD OF '$x = a$'

For some $h > 0$, sufficiently small, let the function $y = f(x)$ be defined in the interval $(a-h, a)$ then it is said that the function $y = f(x)$ is defined in the left-neighbourhood of $x = a$.

Similarly, if the function $y = f(x)$ be defined in the interval $(a, a+h)$ then it is said that the function $y = f(x)$ is defined in the right-neighbourhood of $x = a$.

If the function $y = f(x)$ be defined in left-neighbourhood of $x = a$ or right-neighbourhood of $x = a$, then it is said that the function $y = f(x)$ is defined in the neighbourhood of $x = a$.

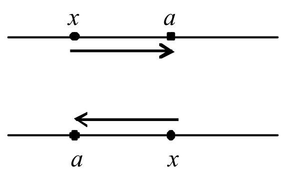
It must be noted here that the value 'a' itself may or may not be included in the domain which is actually not being considered in its neighbourhood.

2. MEANING OF '$x \to a$'

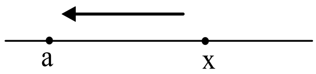
Let x be a variable and 'a' be a constant. If x assumes values nearer and nearer to 'a' but x is strictly smaller than 'a' then this statement is mathematically written as $x \to a^-$.

Similarly, $x \to a^+$, implies that x assumes values nearer and nearer to 'a' but x is strictly greater than 'a'.

In general by 'x tends to a' we mean that
(i) $x \neq a$
(ii) x assumes values nearer and nearer to 'a' and
(iii) we are not specifying the manner in which x should approach to 'a'.
x may approach to 'a' from left or right as shown in figure.

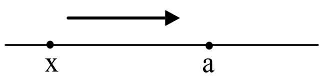


If 'x' approach to 'a' from any point on the right of $x = a$ in the real number line (i.e. the x axis) but it never crosses $x = a$, then it is written as $x \to a^+$.



$\Leftrightarrow$ $x \to a^+$

Similarly,



$\Leftrightarrow$ $x \to a^-$

3. INDETERMINATE FORMS

Some times we come across some functions which do not have definite value corresponding to some particular value of the variable.

For example, the function $f(x) = \dfrac{x^2-4}{x-2}$, converts into $\dfrac{0}{0}$ if $x = 2$ is substituted.
Hence, $f(2)$ cannot be determined. Such a form is called an ***Indeterminate Form***.

There are total *7 **Indeterminate Forms*** given as

(1) $\dfrac{0}{0}$, (2) $\dfrac{\infty}{\infty}$, (3) $\infty - \infty$, (4) $0 \times \infty$, (5) 1^∞, (6) ∞^0, (7) 0^0.

Note : Here 0 and 1 are all approaching values, not the exact values.

EXAMPLE :

Which of the following are forming indeterminate form. Also indicate the form

(i) $\frac{1}{x}$ as $x \to 0$ (ii) $\frac{1-x}{1-x^2}$ as $x \to 1$

(iii) $x \ell n x$ as $x \to 0$ (iv) $\left(\frac{1}{x}-\frac{1}{x^2}\right)$ as $x \to 0$

(v) $(\sin x)^x$ as $x \to 0$ (vi) $(\ell n x)^x$ as $x \to 0$

(vii) $(1+\sin x)^{\frac{1}{x}}$ as $x \to 0$ (viii) $\frac{\sec x}{\tan x}$ as $x \to \frac{\pi}{2}$

Sol. (i) No (ii) $\frac{0}{0}$ form

(iii) $0 \times \infty$ form (iv) $(\infty - \infty)$ form

(v) $(0)^0$ form (vi) $(\infty)^0$ form

(vii) $(1)^\infty$ form (viii) $\frac{\infty}{\infty}$ form

4. LIMIT OF A FUNCTION

DEFINITION 1

Let the function y = f(x) be defined in a certain neighbourhood of a point x = a . The function y = f(x) approaches the limit L ($y \to L$) as x approaches 'a' ($x \to a$). If for every positive number h, arbitrarily small, we are able to indicate k > 0, arbitrarily small, such that for all x, different from 'a' and satisfying the inequality.

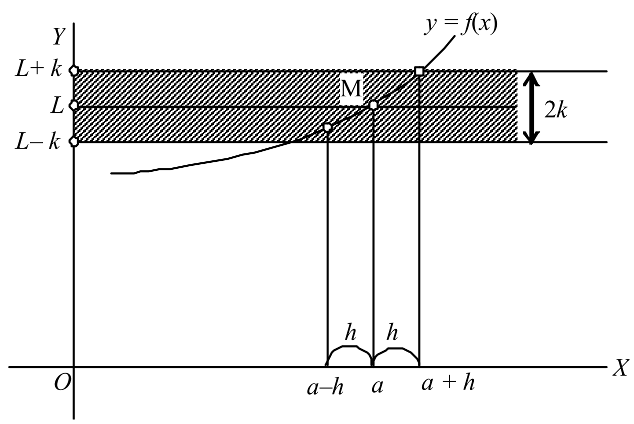
$$|x-a| < h$$

we have the inequality

$$|f(x)-L| < k$$

then $\lim_{x \to a} f(x) = L$

or $f(x) \to L$ as $x \to a$ or limiting value of f(x) is L as $x \to a$.



DEFINITION 2

Let $y = f(x)$ be a function of x and the limiting value of y is required for $x \to a$, then we consider the values of the function at the points which are very near to 'a'.

If these values tend to a definite unique number L as x tends to 'a' (either from left or from right) then this unique number L is called the limit of f(x) at x = a and we write it as $\lim_{x\to a} f(x) = L$

Illustration 1 $\lim_{x\to 2}(x+2)$

Sol. x + 2 being a polynomial in x, its limit as $x \to 2$ is given by $\lim_{x\to 2}(x+2) = 2 + 2 = 4$

Illustration 2 $\lim_{x\to 2} x(x-1)$

Sol. x(x – 1) being a polynomial in x, its limit as $x \to 2$ is given by $\lim_{x\to 2} x(x-1) = 2(2-1) = 2$

Illustration 3 $\lim_{x\to 2}\frac{x^2+4}{x+2}$

Sol. $\lim_{x\to 2}\frac{x^2+4}{x+2} = \frac{(2)^2+4}{2+2} = 2$

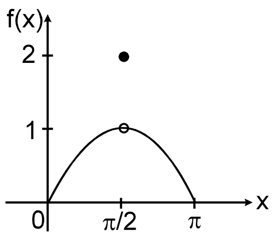
Illustration 4 $\lim_{x\to 0}\cos(\sin x)$

Sol. $\lim_{x\to 0}\cos(\sin x) = \cos\left(\lim_{x\to 0}\sin x\right) = \cos 0 = 1$

Illustration 5 If $f(x) = x^3 + 1$ then $\lim_{x\to 1} f(x) = f(1) = 2$.

Illustration 6

Find $\lim_{x\to \pi/2} f(x)$

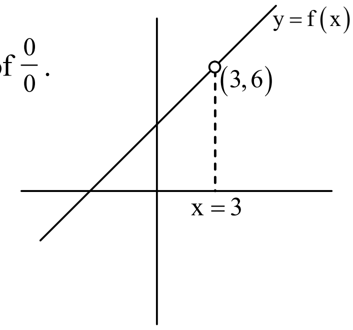


Sol. Here $\lim_{x\to \pi/2} f(x) = 1$

Illustration 7 Find the limiting value of $f(x) = \frac{x^2-9}{x-3}$ as $x \to 3$.

Sol. At x = 3, $f(x) = \frac{x^2-9}{x-3}$ converts into an indeterminate form of $\frac{0}{0}$.

Now when x tends to 3 from left or from right, it can be easily observed from the graph that the value of f(x) tends to 6. Hence



$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)}$$

$$= \lim_{x\to 3}(x+3) = 3 + 3 = 6$$

5. EXISTENCE OF LIMIT

The limit of a function f(x) at a point x = a exists and equals to L if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ = a finite value, L.

Here $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ are called left hand limit (L.H.L.) and right hand limit (R.H.L.) respectively.

Thus, if $\lim_{x\to a} f(x)$ exists then $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ = a finite value, L $\Rightarrow$ L.H.L. = R.H.L. = L

$\Rightarrow \lim_{x\to a} f(x) = L$

Illustration 8 The value of $\lim_{x\to 1} [x]$ is, where [ ] represents the greatest integer function.

(A) 1 (B) 2 (C) 4 (D) Does not exist

Sol. Left hand limit = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} [x] = 0$

and Right hand limit = $\lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} [x] = 1$

$\because \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

$\therefore$ limit does not exist.

Illustration 9 If $f(x) = \begin{cases} \dfrac{1}{1+e^{-1/x}} & x \neq 0 \\ 0, & x = 0 \end{cases}$ then at x = 0

(A) right hand limit of f(x) exists but not left-hand limit
(B) left-hand limit of f(x) exists but not right- hand limit
(C) both limits exists but are not equal
(D) both limits exist and are equal

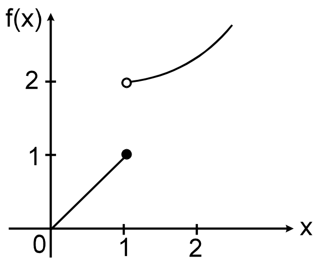
Sol. $f(0^-) = \lim_{h\to 0} \dfrac{1}{1+e^{1/h}} = \dfrac{1}{1+\infty} = 0$

$f(0^+) = \lim_{h\to 0} \dfrac{1}{1+e^{-1/h}} = \dfrac{1}{1+0} = 1$

$\therefore$ Both limits exist but are not equal.

Illustration 10

Find $\lim_{x\to 1} f(x)$



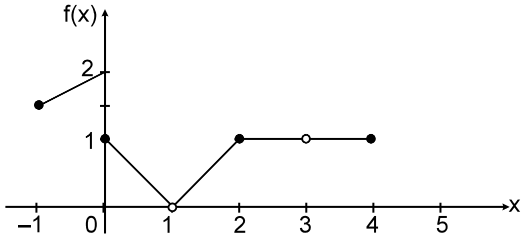
Sol.

Left hand limit = 1 Right hand limit = 2

Hence $\lim_{x\to 1} f(x)$ does not exist.

Illustration 11 From the adjoint graph of y = f(x), find

(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to 1} f(x)$ (iii) $\lim_{x\to 2} f(x)$ (iv) $\lim_{x\to 3} f(x)$ (v) $\lim_{x\to 4} f(x)$



Sol. (i) Here L.H.L. = 2 and R.H.L. = 1

$\therefore$ $\lim_{x\to 0} f(x)$ does not exist

because left hand limit $\neq$ right hand limit.

(ii) $\lim_{x\to 1} f(x) = 0$ (iii) $\lim_{x\to 2} f(x) = 1$

(iv) $\lim_{x\to 3} f(x) = 1$ (v) $\lim_{x\to 4} f(x) = 1$

6. THEOREMS ON LIMITS

Let $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both individually exist and are finite, then

(1) $\lim_{x\to a} [k\, f(x)] = k \lim_{x\to a} f(x)$, where k is a constant

(2) $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(3) $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(4) $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(5) $\lim_{x\to a} [f(x)/g(x)] = [\lim_{x\to a} f(x)] / [\lim_{x\to a} g(x)]$ provided that $\lim_{x\to a} g(x) \neq 0$

(6) $\lim_{x\to a} f[g(x)] = f[\lim_{x\to a} g(x)]$, provided that $\lim_{x\to a} g(x)$ = a finite value k and $\lim_{x\to k} f(x)$ is also finite.

(7) $\lim_{x\to a} [f(x) + k] = \lim_{x\to a} f(x) + k$

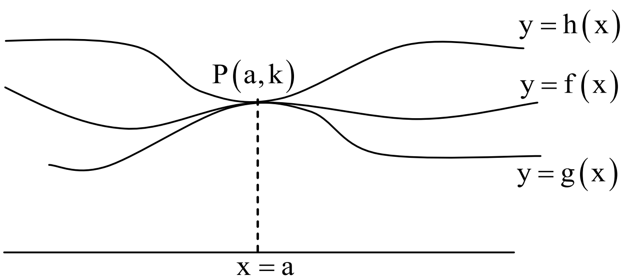
(8) $\lim_{x\to a} [f(x)]^{g(x)} = \{\lim_{x\to a} f(x)\}^{\lim_{x\to a} g(x)}$

(9) $\lim_{x\to +\infty} f(x) \Leftrightarrow \lim_{x\to 0^+} f(1/x)$ and $\lim_{x\to -\infty} f(x) \Leftrightarrow \lim_{x\to 0^-} f(1/x)$ **(Important)**

(10) **Sandwitch theorem or Squeeze Play theorem :**

If $g(x) \leq f(x) \leq h(x)$ in the neighbourhood of x = a such that $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = k$, a finite quantity,

then $\lim_{x\to a} f(x) = k$



7. SOME STANDARD LIMITS

(1) $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{x}{\sin^{-1}x}=1$

Also, $\left[\lim_{x\to 0}\frac{\sin x}{x}\right]=\left[\lim_{x\to 0}\frac{x}{\sin^{-1}x}\right]=0$ and $\left[\lim_{x\to 0}\frac{x}{\sin x}\right]=\left[\lim_{x\to 0}\frac{\sin^{-1}x}{x}\right]=1$

(where [ ] denotes the greatest integer function)

as $0<\frac{\sin x}{x}<1$ and $0<\frac{x}{\sin^{-1}x}<1$

(2) $\lim_{x\to 0}\cos x=\lim_{x\to 0}\left(\frac{1}{\cos x}\right)=1$

(3) $\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{x}{\tan x}=\lim_{x\to 0}\frac{\tan^{-1}x}{x}=\lim_{x\to 0}\frac{x}{\tan^{-1}x}=1$

(4) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{x\to 0}(1+x)^{1/x}=e$

(5) If $\lim_{x\to a}f(x)\to 1$ and $\lim_{x\to a}g(x)\to\infty$ then $\lim_{x\to a}\{f(x)\}^{g(x)}=e^{\lim_{x\to a}(f(x)-1)\cdot g(x)}$

(6) $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a\ (a>0)$ and $\lim_{x\to 0}\frac{e^x-1}{x}=1$

(7) $\lim_{x\to a}\frac{x^n-a^n}{x-a}=n\cdot a^{n-1}$ and $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$

(8) $\lim_{x\to 0}\frac{\ell n(1+x)}{x}=1$ and $\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e$

(9) $\lim_{x\to\infty}a^x=\begin{cases}0 & \text{if } 0\le a<1\\ 1 & \text{if } a=1\\ \infty & \text{if } a>1\\ \text{D.N.E.} & \text{if } a<0\end{cases}$

(10) $\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+.........+a_m}{b_0x^n+b_1x^{n-1}+.........+b_n}=\begin{cases}\frac{a_0}{b_0} & \text{if } m=n\\ 0 & \text{if } m<n\\ \infty & \text{if } m>n\ \&\ a_0\cdot b_0>0\\ -\infty & \text{if } m>n\ \&\ a_0\cdot b_0<0\end{cases}$

8. IMPORTANT EXPANSIONS

(1) $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+............\ x\in R$

(2) $a^x=1+\frac{x\,ln\,a}{1!}+\frac{x^2\,ln^2a}{2!}+\frac{x^3ln^3a}{3!}+.........a>0$

(3) $ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.........\text{for } -1<x\le 1$

(4) $\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots \; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(5) $\cos x = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots x \in (-\pi, \pi)$

(6) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(7) $\sin^{-1} x = x + \frac{1^2.x^3}{3!} + \frac{1^2.3^2.x^5}{5!} + \frac{1^2.3^2.5^2\,x^7}{7!} + \ldots\ldots = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \ldots\ldots$

(8) $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \left(x + \frac{1^2.x^3}{3!} + \frac{1^2.3^2.x^5}{5!} + \frac{1^2.3^2.5^2\,x^7}{7!} + \ldots\ldots\right)$

$$= \frac{\pi}{2} - \left(x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \ldots\ldots\right)$$

(9) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

(10) $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5^2.x^4}{4!} + \frac{61x^6}{6!} + \ldots\ldots$

(11) $\sin h\, x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$

(12) $\cos h\, x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$

(13) $\tan h\, x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots\ldots$

(14) **The Binomial Theorem :**

$(1+x)^n = 1 + n\cdot x + \frac{n\cdot(n-1)}{2!}\cdot x^2 + \frac{n(n-1)(n-2)}{3!}\cdot x^3 + \ldots\ldots$ (for rational values of n only)

(15) $(1+x)^{1/x} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\ldots\ldots\right)$

9. STANDARD APPROACHES TO EVALUATE THE LIMIT OF A FUNCTION

1. Substitution

Illustration 12 Evaluate : $\lim\limits_{x\to\infty} \frac{3x^2 + 5x + 1}{7x^2 + 3x - 1}$

Sol. Put $x = \frac{1}{y}$, $\lim\limits_{x\to\infty} \frac{3x^2+5x+1}{7x^2+3x-1} = \lim\limits_{y\to 0^+} \frac{y^2+5y+3}{y^2+3y+7} = \frac{3}{7}$

Illustration 13 Evaluate : $\lim\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x^3}$

Sol. The given limit $l = \operatorname{Lim}\limits_{x\to 0} \frac{e^x - e^{-x} - 2x}{x^3}$, put $x = 3y$

$$= \lim_{x\to 0} \frac{e^{3y} - e^{-3y} - 6y}{27y^3} = \lim_{x\to 0} \frac{(e^y - e^{-y})^3 + 3(e^y - e^{-y}) - 6y}{27y^3}$$

$$= \lim_{x\to 0} \frac{8}{27}\left(\frac{e^{2y}-1}{2y}\right)^3 + \lim_{x\to 0}\frac{1}{9}\left(e^y - e^{-y} - 2y\right) \qquad \left(\text{Using } \lim_{y\to 0}\frac{e^{2y}-1}{2y} = 1\right)$$

$$= \frac{8}{27} + \frac{1}{9}l = \frac{8l}{9} = \frac{8}{27} \Rightarrow l = \frac{1}{3}$$

Illustration 14 Evaluate : $\lim_{x\to 0}\frac{x - \sin x}{x^3}$

Sol. The given limit $l = \lim_{x\to 0}\frac{x - \sin x}{x^3} = \lim_{x\to 0}\frac{3t - \sin 3t}{27t^3}$

$$= \lim_{x\to 0}\frac{3t - (3\sin t - 4\sin^3 t)}{27t^3} = \lim_{x\to 0}\frac{3(t - \sin t)}{27t^3} + \lim_{x\to 0}\frac{4\sin^3 t}{27t^3} = \frac{1}{9}l + \frac{4}{27}$$

$$\Rightarrow \frac{8l}{9} = \frac{4}{27} \Rightarrow l = \frac{1}{6}$$

Illustration 15 Evaluate the limit : $\lim_{x\to 1}\left(\frac{1}{\ell n\, x} - \frac{1}{x-1}\right)$

Sol. The given limit $\ell = \lim_{x\to 1}\left(\frac{1}{\ell n\, x} - \frac{1}{x-1}\right)$

Put $x = \frac{1}{t}$, then $\ell = \lim_{t\to 1}\left(\frac{1}{\ell n\left(\frac{1}{t}\right)} - \frac{1}{\frac{1}{t} - 1}\right)$

$$= \lim_{t\to 1}\left(\frac{-1}{\ell n\, t} - \frac{t}{1-t}\right)$$

$$= \lim_{t\to 1}\left(\frac{-1}{\ell n\, t} + 1 + \frac{1}{t-1}\right) = 1 - \lim_{t\to 1}\left(\frac{1}{\ell n\, t} - \frac{1}{t-1}\right)$$

$\therefore \quad \ell = 1 - \ell$

$\therefore \quad 2\ell = 1$

$\Rightarrow \quad \ell = \frac{1}{2}$

2 Factorization

Illustration 16 Evaluate : $\lim_{x\to 1}\frac{x^2 - 3x + 2}{x^2 - 1}$

Sol. Note that, for x = 1 both the numerator and the denominator of the given fraction vanish.

Hence, it converts into an indeterminate form of $\frac{0}{0}$.

Therefore we have $\lim_{x\to1}\frac{x^2-3x+2}{x^2-1}=\lim_{x\to1}\frac{(x-1)(x-2)}{(x-1)(x+1)}=\lim_{x\to1}\frac{x-2}{x+1}=-\frac{1}{2}$

Illustration 17 Evaluate : $\lim_{x\to1}\frac{x^3-x^2\log x+\log x-1}{x^2-1}$

Sol. The given limit $=\lim_{x\to1}\frac{(x^3-1)-(x^2-1)\log x}{x^2-1}$

$$=\lim_{x\to1}\frac{(x-1)(x^2+x+1)-(x-1)(x+1)\log x}{(x-1)(x+1)}$$

$$=\lim_{x\to1}\frac{(x-1)\left[x^2+x+1-(x+1)\log x\right]}{(x-1)(x+1)}$$

$$=\frac{1^2+1+1-(1+1)\log 1}{(1+1)}=\frac{3}{2}$$

Illustration 18 Evaluate : $\lim_{x\to2}\left[\frac{1}{x-2}-\frac{2(2x-3)}{x^3-3x^2+2x}\right]$

Sol. The given limit $=\lim_{x\to2}\left[\frac{1}{x-2}-\frac{2(2x-3)}{x(x-1)(x-2)}\right]$

$$=\lim_{x\to2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$=\lim_{x\to2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]=\lim_{x\to2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right]$$

$$=\lim_{x\to2}\left[\frac{x-3}{x(x-1)}\right]=-\frac{1}{2}$$

3 Rationalization or Double Rationalisation

Illustration 19 Evaluate : $\lim_{x\to\pm\infty}x\left(\sqrt{x^2+k}-x\right), k>0$

Sol. The given limit $=\lim_{x\to\pm\infty}x\left(\sqrt{x^2+k}-x\right)\frac{\left(\sqrt{x^2+k}+x\right)}{\left(\sqrt{x^2+k}+x\right)}$

$$=\lim_{x\to\pm\infty}\frac{x\left(x^2+k-x^2\right)}{\left(\sqrt{x^2+k}+x\right)}=\lim_{x\to\pm\infty}\frac{xk}{\left(|x|\sqrt{\left(1+\frac{k}{x^2}\right)}\right)+x}$$

Here we have to consider two cases

(i) When $x\to\infty$; $|x|=x$

then the given limit $=\lim_{x\to\infty}\frac{xk}{x\sqrt{\left(1+\frac{k}{x^2}\right)}+x}=\lim_{x\to\infty}\frac{xk}{x\left(\sqrt{\left(1+\frac{k}{x^2}\right)}+1\right)}=\frac{k}{2}$

(ii) When $x \to -\infty$; $|x| = -x$

then we have $\lim_{x\to-\infty} \dfrac{xk}{-x\sqrt{\left(1+\dfrac{k}{x^2}\right)}+x}$

$$\lim_{x\to-\infty} \frac{xk}{x\left(-\sqrt{1+\dfrac{k}{x^2}}+1\right)} \to \frac{k}{-1^{+}+1} \to \frac{k}{0^{-}} \to -\infty$$

Illustration 20 Evaluate : $\lim_{x\to 0} \dfrac{\sqrt{1+x+x^2}-1}{x}$

Sol. By rationalization of numerator

$$= \lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x} \cdot \frac{\sqrt{1+x+x^2}+1}{\sqrt{1+x+x^2}+1}$$

$$= \lim_{x\to 0} \frac{1+x+x^2-1}{x\left(\sqrt{1+x+x^2}+1\right)}$$

$$= \lim_{x\to 0} \frac{1+x}{\sqrt{1+x+x^2}+1} = \frac{1}{2}$$

Illustration 21 Evaluate : $\lim_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Sol. The given limit taken the form $\dfrac{0}{0}$ when $x \to 0$. Rationalising the numerator, we get

$$= \lim_{x\to 0} \left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x} \times \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \lim_{x\to 0} \left[\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$= \lim_{x\to 0} \left[\frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$= \lim_{x\to 0} \left[\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{2}{2} = 1$$

Illustration 22 $\lim_{x\to 1} \dfrac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$

Sol. The given limit $= \lim_{x\to 1} \left[\dfrac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)(x-1)}\right]$

$$= \lim_{x\to1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}\right]$$

$$= \lim_{x\to1}\left[\frac{2x-3}{(2x+3)\left(\sqrt{x}+1\right)}\right]$$

$$= \frac{-1}{(5)(2)} = \frac{-1}{10}$$

4 Use of Binomial Theorem and other expansions

Illustration 23 f(x) is integral of $\frac{2\sin x-\sin 2x}{x^3}$, $x\neq0$ then, find $\lim_{x\to0} f'(x)$

Sol. $\lim_{x\to0} f'(x) = \lim_{x\to0}\frac{2\sin x-\sin 2x}{x^3}$

$$= \lim_{x\to0}\frac{2\left(x-\frac{x^3}{3!}........\right)-\left(2x-\frac{8x^3}{3!}\right)}{x^3}$$

$$= \frac{8-2}{3!} = 1$$

Illustration 24 Evaluate : $\lim_{x\to0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$

Sol. $\lim_{x\to0}\frac{2x-\left(x+\frac{x^3}{6}+...\right)}{2x+\left(x-\frac{x^3}{3}+...\right)}=\frac{1}{3}$

Illustration 25 Evaluate : $\lim_{x\to0}\frac{e^x-1-x}{x^2}$

Sol. $\lim_{x\to0}\frac{e^x-1-x}{x^2}$

$$= \lim_{x\to0}\frac{\left(1+x+\frac{x^2}{2!}.......\right)-1-x}{x^2} = \frac{1}{2}$$

Illustration 26 Evaluate : $\lim_{x\to0}\frac{(7+x)^{1/3}-2}{x-1}$

Sol. Put x = 1 + h,

$$\lim_{h\to0}\frac{(8+h)^{1/3}-2}{h}$$

$$\lim_{h\to 0}\frac{2.\left(1+\frac{h}{8}\right)^{1/3}-2}{h}=\lim_{h\to 0}\frac{2\left\{1+\frac{1}{3}\cdot\frac{h}{8}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{h}{8}\right)^2}{1.2}+.......-1\right\}}{h}$$ {using binomial theorem}

$$=\lim_{h\to 0} 2\times\frac{1}{24}=\frac{1}{12}$$

Illustration 27 Evaluate : $\lim_{x\to\infty}\frac{x-2}{2x-3}$

Sol. $\lim_{x\to\infty}\frac{x-2}{2x-3}$

$\lim_{x\to\infty}\frac{1-2/x}{2-3/x}=\frac{1}{2}$.

Illustration 28 Evaluate : $\lim_{x\to\infty}\frac{x^2-4x+5}{3x^2-x^3+2}$

Sol. $\lim_{x\to\infty}\frac{x^2-4x+5}{3x^2-x^3+2}$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{4}{x^2}+\frac{5}{x^3}}{\frac{3}{x}-1+\frac{2}{x^3}}=0$$

Illustration 29 Evaluate : $\lim_{x\to-\infty}\frac{\sqrt{3x^2+2}}{x-2}$

Sol. $\lim_{x\to-\infty}\frac{\sqrt{3x^2+2}}{x-2}$

Put $x=\frac{-1}{t}$ as $x\to-\infty$, $t\to 0^+$

$$=\lim_{t\to 0^+}\frac{\sqrt{3+2t^2}\cdot\frac{1}{\sqrt{t^2}}}{\frac{-1-2t}{t}}=\lim_{t\to 0^+}\frac{\sqrt{3+2t^2}}{-(1+2t)}\;\frac{t}{|t|}$$

$$=\frac{\sqrt{3}}{-1}=-\sqrt{3}.$$

Illustration 30 Evaluate : $\text{Lim}_{n\to\infty}\frac{(3(n+1))!}{(n+1)^3(3n)!}$

Sol. $$\text{Lim}_{n\to\infty}\frac{(3n+3)(3n+2)(3n+1)(3n)!}{(n+1)^3(3n)!}=\text{Lim}_{n\to\infty}\frac{27n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{3n}\right)\left(1+\frac{1}{3n}\right)}{n^3\left(1+\frac{1}{n}\right)^3}=27$$

Illustration 31 Let $S_n = 1 + 2 + 3 + ... + n$ and

$$P_n = \frac{S_2}{S_2-1}\cdot\frac{S_3}{S_3-1}\cdot\frac{S_4}{S_4-1}\cdot ....\cdot\frac{S_n}{S_n-1}$$ where $n \in N\ (n \ge 2)$. Find $\operatorname*{Lim}_{n\to\infty} P_n$.

Sol. $S_n = \frac{n(n+1)}{2}$ and $S_n - 1 = \frac{(n+2)(n-1)}{2}$

$$\therefore \quad \frac{S_n}{S_n-1} = \frac{n(n+1)}{2}\cdot\frac{2}{(n+2)(n-1)} \Rightarrow \frac{S_n}{S_n-1} = \left(\frac{n}{n-1}\right)\left(\frac{n+1}{n+2}\right)$$

$$P_n = \left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot..........\cdot\frac{n}{n-1}\right)\left(\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot..........\cdot\frac{n+1}{n+2}\right)$$

$$P_n = \left(\frac{n}{1}\right)\left(\frac{3}{n+2}\right)$$

$$\Rightarrow \operatorname*{Lim}_{n\to\infty} P_n = 3$$

Illustration 32 Evaluate : $\lim_{x\to\infty} x\sin\frac{1}{x}$

Sol. $\lim_{x\to\infty} x\sin\frac{1}{x}$

$$= \lim_{x\to\infty}\frac{\sin(1/x)}{1/x} = 1 \qquad \left(\text{Using } \lim_{y\to 0}\frac{\sin y}{y} = 1\right)$$

IN-CHAPTER EXERCISE 1

1. The value of $\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{1-x}\right)$ is

(A) 0 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) 1

2. The value of $\lim_{x\to 2}\frac{|x-2|}{x-2}$ is

(A) 0 (B) 1 (C) –1 (D) does not exist

3. The value of $\lim_{x\to 0}\frac{1}{3+2^{\frac{1}{x}}}$ is

(A) 0 (B) 1/3 (C) 1/2 (D) does not exist

4. The value of $\lim_{x\to 0}\frac{1+2^{\frac{1}{x}}}{3+2^{\frac{1}{x}}}$ is

(A) 0 (B) 1/3 (C) 1/2 (D) does not exist

5. The value of $\lim_{x\to 3}\frac{x}{[x]}$; where [ ] is the greatest integer function, is

(A) 0 (B) 1 (C) –1 (D) does not exist

6. The value of $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$ is

(A) 0 (B) 1 (C) –1 (D) does not exist

7. The value of $\lim\limits_{n\to\infty} \left(\dfrac{1}{n^2}+\dfrac{2}{n^2}+\dfrac{3}{n^2}+\ldots+\dfrac{n}{n^2}\right)$ is

(A) 0 (B) 1/3 (C) 1/2 (D) 1/4

8. The value of $\lim\limits_{n\to\infty} \dfrac{\sum n^3}{n^4}$ is

(A) 0 (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{1}{2}$

9. The value of $\lim\limits_{n \to \infty} \dfrac{n!}{(n+1)!-n!}$ is

(A) 0 (B) 1 (C) –1 (D) does not exist

10. The value of $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+1} - \sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1} - \sqrt[5]{x^4+1}}$ is

(A) 0 (B) $\dfrac{1}{2}$ (C) $-\dfrac{1}{2}$ (D) 1

11. The value of $\lim\limits_{x \to \infty} \dfrac{30+\ 4\sqrt{x}+(7\sqrt[3]{x})}{2+\sqrt{4x-7}+\sqrt[3]{6x-2}}$ is

(A) 0 (B) 1 (C) 2 (D) $\dfrac{7}{\sqrt[3]{6}}$

12. The value of $\lim\limits_{x \to \infty} (x-\sqrt{x^2+x})$ is

(A) 0 (B) 1/2 (C) –1/2 (D) 1

13. The value of $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{4x^2+1}-1}$ is

(A) –1/4 (B) 1/4 (C) –1/2 (D) 1/2

14. If $\lim\limits_{x \to \infty} \left(\dfrac{x^2+1}{x+1}-ax-b\right) = b$, where a, b are constants, then the value of $a + b$, is

(A) 0 (B) 1/2 (C) –1/2 (D) 1

15. The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1+x^2}}{x}$ is

(A) 0 (B) $\dfrac{1}{2}$ (C) $-\dfrac{1}{2}$ (D) 1

16. The value of $\lim\limits_{x\to0} \dfrac{\sqrt{1+x+x^2}-1}{x}$ is

(A) 0 (B) $\dfrac{1}{2}$ (C) $-\dfrac{1}{2}$ (D) 1

17. The value of $\lim_{x\to1^+} \dfrac{\sqrt{x^2-1}+\sqrt{x-1}}{\sqrt{x^2-1}}$ is

(A) 1 (B) $1-\dfrac{1}{\sqrt{2}}$ (C) $1+\dfrac{1}{\sqrt{2}}$ (D) does not exist

18. The value of $\operatorname{Lim}_{x\to2} \dfrac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$ is

(A) 1 (B) 2 (C) 4 (D) 8

19. The value of $\operatorname{Lim}_{x\to\frac{\pi}{2}} (\sec x - \tan x)$ is

(A) 0 (B) 1 (C) 1/2 (D) -1

20. The value of $\operatorname{Lim}_{n\to\infty} \dfrac{3^{n+1}+2^{n+2}}{5.3^n-2^{n-1}}$ is

(A) $\dfrac{1}{5}$ (B) $\dfrac{2}{5}$ (C) $\dfrac{3}{5}$ (D) 1

ANSWER KEY

1	A	2	D	3	D	4	D	5	D	6	A	7	C	8	B	9	A
10	D	11	C	12	C	13	D	14	B	15	B	16	B	17	C	18	D
19	A	20	C														

5 By application of standard limits

Illustration 33 Evaluate : $\lim_{x\to2} \dfrac{x^3-8}{x^2-4}$

Sol. Method I

$$\frac{x^3-(2)^3}{x^2-(2)^2} = \frac{x^3-(2)^3}{x-2} \div \frac{x^2-(2)^2}{x-2}$$

$$\Rightarrow \lim_{x\to2} \frac{x^3-8}{x^2-4} = \lim_{x\to2} \frac{x^3-(2)^3}{x-2} \div \lim_{x\to2} \frac{x^2-(2)^2}{x-2}$$

$$= 3(2^2) \div 2(2^1) \qquad \left(\text{using } \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right)$$

$$= 12 \div 4 = 3$$

Method II

$$\lim_{x\to2} \frac{x^3-8}{x^2-4} = \lim_{x\to2} \frac{(x^2+2x+4)(x-2)}{(x+2)(x-2)} = \lim_{x\to2} \frac{x^2+2x+4}{x+2}$$

$$= \frac{(2)^2+2(2)+4}{2+2} = \frac{12}{4} = 3$$

Illustration 34 Find $\lim\limits_{x\to -1}\dfrac{1+x^{1/3}}{1+x^{1/5}}$

Sol. Limit $= \lim\limits_{x\to -1}\dfrac{x^{1/3}-(-1)}{x^{1/5}-(-1)} = \dfrac{5}{3}$

Note : $\lim\limits_{x\to -1}\dfrac{(x^{1/5})^5-(-1)}{x^{1/5}-(-1)} = 5(-1)^4$

$\lim\limits_{x\to 1}\dfrac{(x^{1/3})^3-(-1)}{x^{1/3}-(-1)} = 3\,(-1)^2$

Illustration 35 Evaluate : $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

Sol. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x} \Rightarrow \lim\limits_{x\to 0}\dfrac{\sin 2x}{2x}.2 = 2$

Illustration 36 Evaluate : $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x}$

Sol. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x} = \lim\limits_{x\to 0}\left[\dfrac{\sin 2x}{2x}.\dfrac{2x}{3x}.\dfrac{3x}{\sin 3x}\right]$

$= \left[\lim\limits_{2x\to 0}\dfrac{\sin 2x}{2x}\right].\dfrac{2}{3}.\left[\lim\limits_{3x\to 0}\dfrac{3x}{\sin 3x}\right]$

$= 1.\dfrac{2}{3}.\left[\lim\limits_{3x\to 0}\dfrac{\sin 3x}{3x}\right] = \dfrac{2}{3}\times 1 = \dfrac{2}{3}$

Illustration 37 Evaluate : $\underset{x\to 0}{\text{Limit}}\ \dfrac{\tan x-\sin x}{x^3}$

Sol. $\underset{x\to 0}{\text{Limit}}\ \dfrac{\tan x-\sin x}{x^3}$

$= \underset{x\to 0}{\text{Limit}}\ \dfrac{\tan x(1-\cos x)}{x^3} = \lim\limits_{x\to 0}\dfrac{\tan x}{x}\cdot\dfrac{1-\cos x}{x^2} = \underset{x\to 0}{\text{Limit}}\ \dfrac{\tan x\,.\,2\sin^2\frac{x}{2}}{x^3}$

$= \underset{x\to 0}{\text{Limit}}\ \dfrac{\tan x}{x}.\dfrac{1}{2}\left(\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \dfrac{1}{2}.$ $\left(\textbf{Note that : }\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = \dfrac{1}{2}\right)$

Illustration 38 Evaluate : $\lim\limits_{x\to 1^-}\dfrac{1-\sqrt{x}}{(\cos^{-1}x)^2}$

Sol. Put $\cos^{-1}x = y$ and $x\to 1^- \Rightarrow y\to 0$

The given limit $= \lim\limits_{y\to 0}\dfrac{1-\sqrt{\cos y}}{y^2}$

now rationalizing numerator

$= \lim\limits_{y\to 0}\dfrac{(1-\cos y)}{y^2(1+\sqrt{\cos y})}$

$$= \lim_{y\to 0}\frac{1-\cos y}{y^2}\lim_{y\to 0}\frac{1}{1+\sqrt{\cos y}} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

Illustration 39 Evaluate : $\lim\limits_{x\to 0}\dfrac{x^2\cos\frac{1}{x}}{\sin x}$

Sol. The given limit $= \lim\limits_{x\to 0}\dfrac{x\cdot\cos(1/x)\cdot x}{\sin x} = \lim\limits_{x\to 0} x\cos\dfrac{1}{x} = 0 \times$ (a finite quantity between -1 and 1) $= 0$

Illustration 40 $\lim\limits_{x\to 0^+}\dfrac{\cos^{-1}(1-x)}{\sqrt{x}}$

Sol. Put $\cos^{-1}(1-x) = \theta$, then as $x \to 0^+, \theta \to 0^+$

$$\lim_{\theta\to 0^+}\frac{\theta}{\sqrt{1-\cos\theta}} = \lim_{\theta\to 0^+}\frac{\theta}{\sqrt{2}\sin(\theta/2)} = \lim_{\theta\to 0^+}\frac{2(\theta/2)}{\sqrt{2}\sin(\theta/2)} = \sqrt{2}$$

Illustration 41 Evaluate : $\lim\limits_{x\to 0}\dfrac{1-\cos(1-\cos x)}{\sin^4 x}$

Sol. $\lim\limits_{x\to 0}\dfrac{1-\cos\left(2\sin^2\frac{x}{2}\right)}{x^4} \Rightarrow \lim\limits_{x\to 0}\dfrac{1-\cos(1-\cos x)}{(1-\cos x)^2}\cdot\dfrac{(1-\cos x)^2}{x^4} = \lim\limits_{x\to 0}\dfrac{1}{2}\cdot\left(\dfrac{1-\cos x}{x^2}\right)^2 = \dfrac{1}{8}$

Illustration 42 Evaluate : $\lim\limits_{x\to 0}\dfrac{\cos\left(\frac{\pi}{2\cos x}\right)}{\sin(\sin x^2)}$

Sol. $l = \lim\limits_{x\to 0}\dfrac{\cos\left(\frac{\pi}{2\cos x}\right)\cdot\left(\sin x^2\right)}{\sin x^2\cdot\sin(\sin x^2)} = \lim\limits_{x\to 0}\dfrac{\cos\left(\frac{\pi}{2\cos x}\right)}{x^2}\cdot\lim\limits_{x\to 0}\dfrac{x^2}{\sin x^2} = \lim\limits_{x\to 0}\dfrac{\cos\left(\frac{\pi}{2\cos x}\right)}{x^2}$

$$l = \lim_{x\to 0}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2\cos x}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{2\cos x}\right)}\cdot\lim_{x\to 0}\frac{\left(\frac{\pi}{2}-\frac{\pi}{2\cos x}\right)}{x^2}\left(\text{taking }\left(-\frac{\pi}{2}\right)\text{ common}\right)$$

$$= (1)\left(-\frac{\pi}{2}\right)\lim_{x\to 0}\frac{(1-\cos x)}{\cos x}\cdot\frac{1}{x^2} = -\frac{\pi}{2}\cdot\frac{1}{2} = -\frac{\pi}{4}\ \text{Ans.]}$$

Illustration 43 Evaluate : $\lim\limits_{x\to 3}\dfrac{e^x - e^3}{x-3}$

Sol. Put $y = x - 3$. So, as $x \to 3$, $y \to 0$. Thus

$$\lim_{x\to 3}\frac{e^x-e^3}{x-3} = \lim_{y\to 0}\frac{e^{3+y}-e^3}{y} = \lim_{y\to 0}\frac{e^3 . e^y - e^3}{y} = e^3\lim_{y\to 0}\frac{e^y-1}{y} = e^3\ .\ 1 = e^3$$

Illustration 44 Evaluate : $\underset{x\to 0}{\text{Limit}} \frac{e^{3x}-1}{x/2}$

Sol. $\underset{x\to 0}{\text{Limit}} \frac{e^{3x}-1}{x/2} = \underset{x\to 0}{\text{Limit}}\ 2 \times 3\ \frac{e^{3x}-1}{3x} = 6.$

Illustration 45 Evaluate : $\lim_{x\to 0} \frac{x(e^x-1)}{1-\cos x}$

Sol. $\lim_{x\to 0} \frac{x(e^x-1)}{1-\cos x} = \lim_{x\to 0} \frac{x^2}{1-\cos x}\cdot\frac{e^x-1}{x} = \frac{1}{\left(\frac{1}{2}\right)}\cdot 1 = 2$

Illustration 46 Evaluate : $\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2}$

Sol. $\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0} \frac{\sin\{\pi(1-\sin^2 x)\}}{x^2}$

$= \lim_{x\to 0} \frac{\sin(\pi - \pi\sin^2 x)}{x^2}$

$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{x^2}$

$= \lim_{x\to 0}\left\{\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\frac{\pi}{1}\times\frac{\sin^2 x}{x^2}\right\}$

$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\pi\times\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 1 \times \pi \times 1 = \pi$

Illustration 47 Evaluate : $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

Sol. $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^{\lim_{x\to\infty}\frac{2}{x}.x} = e^2.$

Illustration 48 Evaluate : $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$

Sol. $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}} = \lim_{x\to 0}\left\{\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right\}^{\frac{1}{x}} = \lim_{x\to 0}\left\{\frac{1+\tan x}{1-\tan x}\right\}^{\frac{1}{x}} = \lim_{x\to 0}\left\{1+\frac{2\tan x}{1-\tan x}\right\}^{\frac{1}{x}}$

$= e^{\lim_{x\to 0}\left\{\frac{2\tan x}{1-\tan x}\right\}\frac{1}{x}}$

$= e^{\lim_{x\to 0} 2\frac{\tan x}{x}.\frac{1}{1-\tan x}} = e^2$

Illustration 49 Evaluate : $\lim_{x\to e}\frac{\ln x-1}{x-e}$

Sol. The given limit $=\lim_{x\to e}\frac{\ln\left(\frac{x}{e}\right)}{e\left(\frac{x}{e}-1\right)}$

Put $\frac{x}{e}-1=y$, as $x\to e, y\to 1$

$\therefore$ The given limit $=\lim_{y\to 0}\frac{\ln(1+y)}{ey}=\frac{1}{e}$

6 By using Sandwich Theorem

Illustration 50 Evaluate : $\lim_{x\to\infty}\frac{[x]}{x}$

Sol. $x-1<[x]\le x,\ \Rightarrow 1-\frac{1}{x}<\frac{[x]}{x}\le 1$ $\{\text{As } x\to\infty \therefore x>0\}$

Now $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)=1.$

Therefore by Sandwitch theorem $\lim_{x\to\infty}\frac{[x]}{x}=1$

Illustration 51 Evaluate : $\mathrm{Lim}_{x\to 0}\, x^3\cos\frac{2}{x}$

Sol. $-1\le\cos\frac{2}{x}\le 1;\ -x^3\le x^3\cos\frac{2}{x}\le x^3$ for $x>0$ and $x^3\le x^3\cos\frac{2}{x}\le -x^3$ for $x<0$

in both the cases limit is zero

Illustration 52 Evaluate : $\lim_{x\to\infty}\frac{5x^2-\sin 3x}{x^2+10}$

Sol. $\because -1\le\sin 3x\le 1$

$\therefore \frac{5x^2-1}{x^2+10}\le\frac{5x^2-\sin 3x}{x^2+10}\le\frac{5x^2+1}{x^2+10}$

$\therefore \lim_{x\to\infty}\frac{5x^2-1}{x^2+10}\le\lim_{x\to\infty}\frac{5x^2-\sin 3x}{x^2+10}\le\lim_{x\to\infty}\frac{5x^2+1}{x^2+10}$

Since $\lim_{x\to\infty}\frac{5x^2-1}{x^2+10}=5$ and $\lim_{x\to\infty}\frac{5x^2+1}{x^2+10}=5$. Hence $\lim_{x\to\infty}\frac{5x^2-\sin 3x}{x^2+10}=5$.

Illustration 53 Evaluate : $\mathrm{Limit}_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+....+\frac{n}{n^2+n}\right)$

Sol. Let $f(n)=\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+....+\frac{n}{n^2+n}$

note that f(n) has n terms which are decreasing

Suppose $h(n) = \left(\frac{n}{n^2+1} + \frac{n}{n^2+1} + \frac{n}{n^2+1} + ... + \frac{n}{n^2+1}\right)$, n terms

$h(n) = \frac{n^2}{n^2+1}$ (obviously $f(n) < h(n)$)

and $g(n) = \left(\frac{n}{n^2+n} + \frac{n}{n^2+n} + \frac{n}{n^2+n} + ... + \frac{n}{n^2+n}\right)$, n terms

$= \frac{n^2}{n^2+n}$ (obviously $g(n) < f(n)$)

Hence $g(n) < f(n) < h(n)$

Since $\underset{n\to\infty}{\text{Limit}}\ g(n) = 1 = \underset{n\to\infty}{\text{Limit}}\ h(n)$

Hence, using Sandwich Theorem $\underset{n\to\infty}{\text{Limit}}\ f(n) = 1$

Illustration 54 If [x] denotes the integral part of x, then evaluate

$$\lim_{n\to\infty} \frac{[1^2x] + [2^2x] + ... + [n^2x]}{n^3}$$

Sol. Let $S_n = [1^2x] + [2^2x] + . . . + [n^2x]$

$x - 1 < [x] \le x$

$\therefore$ $1^2 x - 1 < [1^2x] \le 1^2x$

$2^2x - 1 < [2^2x] \le 2^2x$

$3^2x - 1 < [3^2x] \le 3^2 x$

. . . .

. . . .

$n^2x - 1 < [n^2x] \le n^2x$

$\therefore$ $(1^2 + 2^2 + . . . + n^2) x - n < S_n \le (1^2 + 2^2 + . . . + n^2) x$

$\Rightarrow$ $\frac{n(n+1)(2n+1)}{6n^3}x - \frac{n}{n^3} < \frac{S_n}{n^3} \le \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}.x$

$\Rightarrow$ $\lim_{n\to\infty}\left\{\frac{n(n+1)(2n+1)}{6n^3}x - \frac{1}{n^2}\right\} < \lim_{n\to\infty}\frac{S_n}{n^3} \le \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}.x$

$\because$ $\lim_{n\to\infty}\left\{\frac{n(n+1)(2n+1)}{6n^3}x - \frac{1}{n^2}\right\} = \frac{x}{3}$ and $\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}.x = \frac{x}{3}$

Hence required limit i.e. $\lim_{n\to\infty}\frac{S_n}{n^3} = \frac{x}{3}$.

IN-CHAPTER EXERCISE 2

1. The value of $\underset{x\to 0}{\text{Lim}}\ \frac{\sin 5x - \sin 3x}{\sin x}$ is

(A) 0 (B) 1 (C) 2 (D) does not exist

2. The value of $\underset{x\to 0}{\text{Lim}}\ \frac{\tan 2x - x}{3x - \sin x}$ is

(A) 0 (B) 1/2 (C) –1/2 (D) 1

3. The value of $\lim\limits_{x \to 0} \frac{e^{\sin x} - 1}{x}$ is

(A) 0 (B) 1 (C) –1 (D) None of these

4. The value of $\lim\limits_{x \to 0} \frac{\sin(2 + x) - \sin(2 - x)}{x}$ is

(A) $\sin 2$ (B) $\cos 2$ (C) $2\sin 2$ (D) $2\cos 2$

5. The value of $\lim\limits_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sin^{-1} x}$ is

(A) 0 (B) 1/2 (C) 1 (D) 2

6. The value of $\lim\limits_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{\pi - 4x}$ is

(A) $\frac{1}{\sqrt{2}}$ (B) $-\frac{1}{\sqrt{2}}$ (C) $\frac{1}{2\sqrt{2}}$ (D) $-\frac{1}{2\sqrt{2}}$

7. The value of $\lim\limits_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$ is

(A) 0 (B) 1 (C) 1/2 (D) –1/2

8. The value of $\lim\limits_{x \to 0} \frac{x \ell n(1+x)}{1 - \cos x}$ is

(A) 0 (B) 1 (C) 2 (D) does not exist

9. The value of $\lim\limits_{x \to 0} \frac{a^x - b^x}{x}$ is

(A) $\ell n\left(\frac{a}{b}\right)$ (B) $\ell n\left(\frac{b}{a}\right)$ (C) $\ell n\left(\frac{a}{b}\right)^2$ (D) $\ell n\left(\frac{b}{a}\right)^2$

10. The value of $\lim\limits_{x \to 1} \frac{2.3^x - 3.2^x}{x - 1}$ is

(A) $6\ell n\frac{3}{2}$ (B) $6\ell n\frac{2}{3}$ (C) $3\ell n\frac{3}{2}$ (D) $3\ell n\frac{2}{3}$

11. The value of $\lim\limits_{x \to 0} \frac{a^x + b^x + c^x - 3}{x}$ $(a, b, c > 0)$, is

(A) $\ell n(abc)$ (B) $\ell n(abc)^{-1}$ (C) $\ell n(abc)^{1/3}$ (D) $\ell n(abc)^{-1/3}$

12. The value of $\lim\limits_{x \to 1} \frac{x + x^2 + ... + x^n - n}{x - 1}$ is

(A) $\frac{n}{2}$ (B) $\frac{n+1}{2}$ (C) $\frac{n(n+1)}{2}$ (D) $\frac{n(n+1)}{4}$

13. The value of $\lim_{x\to\frac{1}{\sqrt{2}}} \frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)}$ is

(A) $1/\sqrt{2}$ (B) $-1/\sqrt{2}$ (C) 1 (D) does not exist

14. The value of $\lim_{x\to\infty} x^2 \sin\left(\ell n\sqrt{\cos\frac{\pi}{x}}\right)$ is

(A) $-\frac{\pi^2}{2}$ (B) $-\frac{\pi^2}{4}$ (C) $\frac{\pi^2}{2}$ (D) $\frac{\pi^2}{4}$

15. The value of $\operatorname{Lim}_{x\to 0} (1+\sin x)^{\cot x}$ is

(A) 1 (B) e (C) e^{-1} (D) $e^{-1/2}$

16. The value of $\operatorname{Lim}_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$ is

(A) 1 (B) e (C) $\sqrt{e}$ (D) $\sqrt[3]{e}$

17. The value of $\operatorname{Lim}_{x\to\infty} \left[\frac{2x^2+3}{2x^2+5}\right]^{8x^2+3}$ is

(A) e^{-2} (B) e^{-4} (C) e^{-8} (D) e^{-1}

18. If $\operatorname{Lim}_{x\to\infty} \left(\frac{x+c}{x-c}\right)^x = 4$, then the value of c is

(A) 1 (B) $\ell n2$ (C) $-\ell n2$ (D) $2\ell n2$

19. The value of $\operatorname{Lim}_{x\to 0} \left(\frac{x-1+\cos x}{x}\right)^{\frac{1}{x}}$ is

(A) 1 (B) e (C) $\sqrt{e}$ (D) $1/\sqrt{e}$

20. The value of $\operatorname{Lim}_{n\to\infty} \frac{[x]+[2x]+...+[nx]}{n^2}$, where [ ] is the step function (x is real), is

(A) 0 (B) x (C) x/2 (D) does not exist

ANSWER KEY

1	C	2	B	3	B	4	D	5	C	6	C	7	C
8	C	9	A	10	A	11	A	12	C	13	B	14	B
15	B	16	D	17	C	18	B	19	D	20	C		

SOLVED OBJECTIVE EXAMPLES

1. $\lim_{x\to\frac{\pi}{2}} \frac{\cos 2x+1}{\sin 2x} =$

(a) 0 (b) 1 (c) 2 (d) 1/2

Sol. (a)

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos 2x+1}{\sin 2x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos 2x+1}{\sin 2x} \times \frac{1-\cos 2x}{1-\cos 2x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\sin 2x}{1-\cos 2x} = 0.$$

2. Value of $\lim_{x\to\frac{\pi}{4}} \frac{\sqrt{1-\sin 2x}}{4x-\pi}$ is

(a) 0 (b) 1 (c) ∞ (d) limit does not exist

Sol. (d)

$$L = \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{1-\sin 2x}}{4x-\pi} = \lim_{x\to\frac{\pi}{4}} \frac{\sqrt{(\sin x-\cos x)^2}}{4x-\pi} \Rightarrow L = \lim_{x\to\frac{\pi}{4}} \frac{|\sin x-\cos x|}{4x-\pi}$$

Now

$$LHL = -\lim_{x\to\frac{\pi}{4}^-} \frac{\sin x-\cos x}{4x-\pi} \Rightarrow LHL = -\frac{\sqrt{2}}{4} \lim_{x\to\frac{\pi}{4}^-} \frac{\sin\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}} = -\frac{1}{2\sqrt{2}} \;\&$$

$$RHL = \lim_{x\to\frac{\pi}{4}^+} \frac{\sin x-\cos x}{4x-\pi} \Rightarrow RHL = \frac{\sqrt{2}}{4} \lim_{x\to\frac{\pi}{4}^+} \frac{\sin\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}$$

As $LHL \neq RHL$, hence limit dos not exist.

3. $\lim_{x\to 0} \frac{\ln\left(\tan\left(\frac{\pi}{4}+x\right)\right)}{x} =$

(a) 0 (b) 1 (c) 2 (d) limit does not exist

Sol. (c)

$$L = \lim_{x\to 0} \frac{\ln\left(\tan\left(\frac{\pi}{4}+x\right)\right)}{x} = \lim_{x\to 0} \frac{\ln\left(\frac{1+\tan x}{1-\tan x}\right)}{x} \Rightarrow L = \lim_{x\to 0} \frac{\ln(1+\tan x)-\ln(1-\tan x)}{x}$$

$$\Rightarrow L = \lim_{x\to 0}\left(\frac{\ln(1+\tan x)}{\tan x} + \frac{\ln(1-\tan x)}{-\tan x}\right) \times \frac{\tan x}{x} = 2.$$

4. Value of $\lim\limits_{x\to 2}\dfrac{\sqrt{x}-\sqrt{4}}{\sqrt{x+7}-\sqrt{2x+5}}$ is

(a) $\dfrac{\sqrt{2}}{3}$ (b) $-\dfrac{\sqrt{2}}{3}$ (c) 0 (d) limit does not exist

Sol. (b)

$$L=\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x+7}-\sqrt{2x+5}} \Rightarrow L=\lim_{x\to 2}\frac{x-2}{(x+7)-(2x+5)}\times\frac{\sqrt{x+7}+\sqrt{2x+5}}{\sqrt{x}+\sqrt{2}}$$

$$\Rightarrow L=\lim_{x\to 2}\frac{x-2}{2-x}\times\frac{\sqrt{x+7}+\sqrt{2x+5}}{\sqrt{x}+\sqrt{2}}=-\frac{\sqrt{2}}{3}.$$

5. Value of $\lim\limits_{x\to 2}\dfrac{(x+2)^n-4^n}{\sin \pi x}$ is

(a) $\dfrac{n.4^{n-1}}{\pi}$ (b) $\dfrac{n.4^{n}}{\pi}$ (c) 0 (d) ∞

Sol. (a)

Let $x=h+2$, then

$$L=\lim_{x\to 2}\frac{(x+2)^n-4^n}{\sin \pi x}=\lim_{h\to 0}\frac{(h+4)^n-4^n}{\sin(2\pi+\pi h)}$$

$$\Rightarrow L=\lim_{h\to 0}\frac{h^n+{}^nC_1.h^{n-1}.4+{}^nC_2.h^{n-2}.4^2+\ldots+{}^nC_{n-2}.h^2.4^{n-2}+{}^nC_{n-1}.h.4^{n-1}}{\sin \pi h}$$

$$\Rightarrow L=\lim_{h\to 0}\frac{h\left(h^{n-1}+{}^nC_1.h^{n-2}.4+{}^nC_2.h^{n-3}.4^2+\ldots+{}^nC_{n-2}.h.4^{n-2}+{}^nC_{n-1}.4^{n-1}\right)}{\pi h\left(\dfrac{\sin \pi h}{\pi h}\right)}=\frac{n4^{n-1}}{\pi}.$$

6. Value of $\lim\limits_{x\to 1/2}\dfrac{a^{\sin \pi x-1}-1}{\tan^2 2\pi x}$ is

(a) $-\dfrac{\ln a}{4}$ (b) $\dfrac{\ln a}{4}$ (c) $\dfrac{\ln a}{8}$ (d) $-\dfrac{\ln a}{8}$

Sol. (d)

Let x = 1 + h, then

$$L=\lim_{x\to 1/2}\frac{a^{\sin \pi x-1}-1}{\tan^2 2\pi x}=\lim_{h\to 0}\frac{a^{\cos \pi h-1}-1}{\tan^2 2\pi h}$$

$$=\lim_{h\to 0}\left(\frac{a^{-2\sin^2\frac{\pi h}{2}}-1}{-2\sin^2\dfrac{\pi h}{2}}\right)\times\frac{-2\sin^2\dfrac{\pi h}{2}}{\sin^2 2\pi h}\times\cos^2 2\pi h$$

$$= \ln a \times \lim_{h\to 0} \frac{-2\sin^2 \frac{\pi h}{2}}{\left(4\sin\frac{\pi h}{2}\cos\frac{\pi h}{2}\cos \pi h\right)^2} \times \cos^2 2\pi h$$

$$\Rightarrow L = -\frac{\ln a}{8}.$$

7. Value of $\lim_{x\to\pi^+} (\pi - x)^{\sin x}$ is

(a) 1 (b) 0 (c) π (d) not defined

Sol. (a)

Let $x = \pi - h$, then

$$L = \lim_{h\to 0^+} (h)^{\sinh} \Rightarrow \ln L = \lim_{h\to 0^+} \sin h(\ln h)$$

$$\Rightarrow \ln L = \lim_{h\to 0^+} \frac{\sin h}{h}(h \times \ln h) = 0$$

$$\Rightarrow L = 1 \cdot$$

8. Value of $\lim_{x\to\infty}\left(\frac{x^2+1}{x^2+2}\right)^{2x^2+3}$ is

(a) e (b) e^{-2} (c) e^2 (d) not defined

Sol. (b)

$$L = \lim_{x\to\infty}\left(\frac{x^2+1}{x^2+2}\right)^{2x^2+3} \Rightarrow L = e^{\lim_{x\to\infty}\left(\frac{x^2+1}{x^2+2}-1\right)\left(2x^2+3\right)}$$

$$\Rightarrow L = e^{-\lim_{x\to\infty}\left(\frac{2x^2+3}{x^2+2}\right)}$$

$$\Rightarrow L = e^{-\lim_{x\to\infty}\left(\frac{2+\frac{3}{x^2}}{1+\frac{2}{x^2}}\right)} = e^{-2}.$$

9. Value of $\lim_{x\to 0}\frac{\tan 2x - 2x}{\sin 3x - 3x}$ is

(a) $-\frac{2}{81}$ (b) $\frac{2}{27}$ (c) $-\frac{2}{27}$ (d) $-\frac{4}{27}$

Sol. (c)

$$L = \lim_{x\to 0}\frac{\tan 2x - 2x}{\sin 3x - 3x} \Rightarrow L = \lim_{x\to 0}\frac{\left(2x + \frac{x^3}{3} + \ldots\right) - 2x}{\left(3x - \frac{27x^3}{6} + \ldots\right) - 3x}$$

$$\Rightarrow L=\lim_{x\to 0}\frac{\frac{x^3}{3}+\ldots}{-\frac{27x^3}{6}+\ldots}=-\frac{4}{27}$$

10. Value of $\lim_{x\to 1}\frac{\int_1^{x^2}\sin\left(1-\sqrt{t}\right)}{3x-x^3}$ is

(a) $-\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $-\frac{1}{3}$ (d) $\frac{2}{3}$

Sol. (b)

$$L=\lim_{x\to 1}\frac{\int_1^{x^2}\sin\left(1-\sqrt{t}\right)}{3x-x^3}\Rightarrow L=\lim_{x\to 1}\frac{2x\sin(1-x)}{3\left(1-x^2\right)}$$

$$\Rightarrow L=\lim_{x\to 1}\frac{\sin(1-x)}{(1-x)}\frac{2x}{3(1+x)}=\frac{1}{3}$$

11. Value of $\lim_{x\to 0}\frac{e^{ax}-b\sin x-c}{x^2}=2,(a>0)$, then $a=$

(a) 2 (b) 1 (c) 1/2 (d) 0

Sol. (a)

$$\lim_{x\to 0}\frac{e^{ax}-b\sin x-c}{x^2}=\lim_{x\to 0}\frac{\left(1+ax+\frac{a^2x^2}{2}+\ldots\right)-b\left(x-\frac{x^3}{6}+\ldots\right)-c}{x^2}$$

$$=\lim_{x\to 0}\frac{1-c+(a-b)x+\frac{a^2x^2}{2}+\ldots}{x^2}$$

Now for $L=2,\ 1-c=0,\ a-b=0,\ \frac{a^2}{2}=2$.

Hence $a=b=2$ & $c=1$.

12. If $f(x)=\sqrt{\frac{x-\sin x}{x+\cos^2 x}}$, then $\lim_{x\to\infty}f(x)$ is

(a) 0 (b) ∞ (c) 1 (d) Does not exist

Sol. (c)

$$\lim_{x\to\infty}\sqrt{\frac{x-\sin x}{x+\cos^2 x}}=\lim_{x\to\infty}\sqrt{\frac{1-\frac{\sin x}{x}}{1+\frac{\cos^2 x}{x}}}=1$$

13. If $\lim_{x\to\infty}\frac{a\tan x-\sin bx}{x^2\ln(1+x)}=\frac{11}{2}$, then a + b =

(a) 5/2 (b) 6 (c) 9/2 (d) 3/2

Sol. (b)

$$\lim_{x\to\infty}\frac{a\tan x-\sin bx}{x^2\ln(1+x)}=\lim_{x\to\infty}\frac{a\left(x+\frac{x^3}{3}+...\right)-\left(bx-\frac{b^3x^3}{6}+...\right)}{\left(x^3-\frac{x^4}{2}+...\right)}$$

$$=\lim_{x\to\infty}\frac{(a-b)x+\left(\frac{a}{3}+\frac{b^3}{6}\right)x^3+...}{\left(x^3-\frac{x^4}{2}+...\right)}$$

Now $a=b$, $\frac{a}{3}+\frac{b^3}{6}=\frac{11}{2}$, hence $b^3+2b=33$ or $b=3$.

14. Value of $\lim_{n\to\infty}\left(\frac{1^3+2^3+3^3+...+n^3}{n^4}\right)$ is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) ∞

Sol. (b)

$$\lim_{n\to\infty}\left(\frac{1^3+2^3+3^3+...+n^3}{n^4}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+...\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^2$$

$$=\int_0^1 x^2dx=\frac{1}{3}$$

15. If $f(x)=\lim_{n\to\infty}\frac{x^{2n}\sin x+\cos x}{x^{2n}+2}$, then $f\left(\frac{\pi}{4}\right)+f\left(\frac{3\pi}{4}\right)=$

(a) $\sqrt{2}$ (b) 0 (c) ∞ (d) $\frac{3}{2\sqrt{2}}$

Sol. (d)

If $x<1$, then $\lim_{n\to\infty}x^{2n}=0$, hence $f(x)=\lim_{n\to\infty}\frac{x^{2n}\sin x+\cos x}{x^{2n}+2}\Rightarrow f(x)=\frac{\cos x}{2}$

$\therefore f\left(\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$

If $x>1$, then $\lim_{n\to\infty}\frac{1}{x^{2n}}=0$, hence $f(x)=\lim_{n\to\infty}\frac{\sin x+\frac{\cos x}{x^{2n}}}{1+\frac{2}{x^{2n}}}\Rightarrow f(x)=\sin x$

$\therefore f\left(\frac{3\pi}{4}\right)=\frac{1}{\sqrt{2}}$

Hence $f\left(\frac{\pi}{4}\right)+f\left(\frac{3\pi}{4}\right)=\frac{3}{2\sqrt{2}}$

EXERCISE - 1 (A)

1. If $f(x)=\begin{cases} x, \text{ when } x>1 \\ x^2, \text{ when } x<1 \end{cases}$, then $\lim_{x\to 1} f(x) =$

(a) x^2 (b) x (c) – 1 (d) 1

2. $\lim_{x\to 2}\frac{|x-2|}{x-2}=$

(a) 1 (b) –1 (c) 0 (d) Does not exist

3. If $f(x)=\begin{cases} \frac{2}{5-x}, \text{ when } x<3 \\ 5-x, \text{ when } x>3 \end{cases}$, then

(a) $\lim_{x\to 3^+} f(x)=0$ (b) $\lim_{x\to 3^-} f(x)=0$

(c) $\lim_{x\to 3^+} f(x)\neq \lim_{x\to 3^-} f(x)$ (d) None of these

4. Let the function f be defined by the equation $f(x)=\begin{cases} 3x, & \text{if } 0\le x\le 1 \\ 5-3x, & \text{if } 1<x\le 2 \end{cases}$, then

(a) $\lim_{x\to 1} f(x)=f(1)$ (b) $\lim_{x\to 1} f(x)=3$

(c) $\lim_{x\to 1} f(x)=2$ (d) $\lim_{x\to 1} f(x)$ does not exist

5. $\lim_{x\to 0}\frac{|x|}{x}=$

(a) 1 (b) –1 (c) 0 (d) Does not exist

6. $\lim_{x\to 1}(3x^2+4x+5)=$

(a) 12 (b) –12 (c) 4 (d) Does not exist

7. The value of $\lim_{x\to 2}\frac{3^{x/2}-3}{3^x-9}$ is

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\ln 3$

8. The value of $\lim_{x\to a}\frac{x^5-a^5}{x-a}$ is

(a) 0 (b) $5a^4$ (c) $4a^5$ (d) 1

9. $\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right]$ equals

(a) $\frac{1}{2x}$ (b) $-\frac{1}{2x}$ (c) $\frac{1}{x^2}$ (d) $-\frac{1}{x^2}$

10. The value of $\lim\limits_{x\to 0}\dfrac{\sqrt{1-x^2}-\sqrt{1+x^2}}{x^2}$ is

(a) 1 (b) – 1 (c) – 2 (d) 0

11. $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ equals

(a) 1 (b) $\dfrac{3}{2}$ (c) $\dfrac{1}{4}$ (d) None of these

12. $\lim\limits_{x\to\infty}\dfrac{ax^2+bx+c}{dx^2+ex+f}=$

(a) $\dfrac{b}{e}$ (b) $\dfrac{c}{f}$ (c) $\dfrac{a}{d}$ (d) $\dfrac{d}{a}$

13. $\lim\limits_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$ is equal to

(a) 0 (b) $\dfrac{1}{2}$ (c) $\ln 2$ (d) e^4

14. $\lim\limits_{x\to 1}x^x=$

(a) 1 (b) ∞ (c) Not defined (d) None of these

15. $\lim\limits_{x\to 1}(1+x)^{\frac{1}{x}}=$

(a) 2 (b) e (c) Not defined (d) None of these

16. $\lim\limits_{x\to\frac{\pi}{6}}\left[\dfrac{3\sin x-\sqrt{3}\cos x}{6x-\pi}\right]$

(a) $\sqrt{3}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $-\dfrac{1}{\sqrt{3}}$

17. The value of the limit of $\dfrac{x^3-x^2-18}{x-3}$ as x tends to 3 is

(a) 3 (b) 9 (c) 18 (d) 21

18. The value of the limit of $\dfrac{x^3-8}{x^2-4}$ as x tends to 2 is

(a) 3 (b) (c) 1 (d) 0

19. $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-\sqrt{1-x}}$ is equal to

(a) $\dfrac{1}{2}$ (b) 2 (c) 1 (d) 0

20. $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ equals

(a) $\dfrac{2}{3\sqrt{3}}$ (b) $\dfrac{2}{3\sqrt{3}}$ (c) 0 (d) None of these

21 $\lim\limits_{n\to\infty}\dfrac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}}=$

(a) $\dfrac{99}{100}$ (b) $\dfrac{1}{100}$ (c) $\dfrac{1}{99}$ (d) $\dfrac{1}{101}$

22 The value of $\lim\limits_{x\to 7}\dfrac{2-\sqrt{x-3}}{x^2-49}$ is

(a) $\dfrac{2}{9}$ (b) $-\dfrac{2}{49}$ (c) $\dfrac{1}{56}$ (d) $-\dfrac{1}{56}$

23 $\lim\limits_{n\to\infty}\left[\dfrac{\sum\limits_{r=1}^{n} r^2}{n^3}\right]=$

(a) $-\dfrac{1}{6}$ (b) $\dfrac{1}{6}$ (c) $\dfrac{1}{3}$ (d) $-\dfrac{1}{3}$

24 $\lim\limits_{x\to 0}\dfrac{4^x-9^x}{x(4^x+9^x)}=$

(a) $\log\left(\dfrac{2}{3}\right)$ (b) $\dfrac{1}{2}\log\left(\dfrac{3}{2}\right)$ (c) $\dfrac{1}{2}\log\left(\dfrac{3}{2}\right)$ (d) $\log\left(\dfrac{3}{2}\right)$

25 If $f(x)=\dfrac{2}{x-3}$, $g(x)=\dfrac{x-3}{x+4}$ and $h(x)=-\dfrac{2(2x+1)}{x^2+x-12}$ then $\lim\limits_{x\to 3}\left(f(x)+g(x)+h(x)\right)$ is

(a) -2 (b) -1 (c) $-\dfrac{2}{7}$ (d) 0

26. $\lim\limits_{x\to 0}\dfrac{\sin x+\log(1-x)}{x^2}$ is equal to

(a) 0 (b) $\dfrac{1}{2}$ (c) $-\dfrac{1}{2}$ (d) None of these

27. If $\lim\limits_{x\to\infty}\left[\dfrac{x^3+1}{x^2+1}-(ax+b)\right]=2,$ then

(a) $a=1, b=1$ (b) $a=1,\ b=-1$ (c) $a=1,\ b=-2$ (d) $a=1, b=2$

28. $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+.....+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to

(a) 0 (b) 1 (c) 10 (d) 100

29. $\lim_{x\to 0}\frac{xe^{x}-\log(1+x)}{x^{2}}$ equals

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

30. $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^{3}}$ is equal to

(a) 0 (b) 1 (c) – 1 (d) $\frac{1}{2}$

31. $\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{1-\sin x}}{2x-\pi}$

(a) Exists and it equal $\frac{1}{\sqrt{2}}$ (b) Exists and it equals $-\frac{1}{\sqrt{2}}$

(c) Does not exist (d) None of these

32. $\lim_{x\to 0}\frac{(1-\cos 2x)\ \sin 5x}{x^{2}\sin 3x}=$

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$ (c) $\frac{6}{5}$ (d) $\frac{5}{6}$

33. $\lim_{x\to 0}\frac{x^{3}}{\sin x^{2}}=$

(a) 0 (b) $\frac{1}{3}$ (c) 3 (d) $\frac{1}{2}$

34. $\lim_{x\to 0}\frac{\sin 3x+\sin x}{x}=$

(a) $\frac{1}{3}$ (b) 3 (c) 4 (d) $\frac{1}{4}$

35. $\lim_{x\to 0}\frac{(1+x)^{8}-1-8x}{x^{2}}=$

(a) 28 (b) 56 (c) –28 (d) 16

36. If $f(x)=\begin{cases}\frac{\sin[x]}{[x]}, & x<0\\ 0\ , & 0\le x<1\end{cases}$, then $\lim_{x\to 0}f(x)$ equals

(a) 1 (b) 0 (c) –1 (d) Does not exist

37. Value of $\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3}$ is

(a) 4 (b) –4 (c) $\frac{16}{3}$ (d) 2

38. If $f(x)=\frac{\sin(e^{x-2}-1)}{\log(x-1)}$, then $\lim_{x\to 2}f(x)$ is given by

(a) – 2 (b) –1 (c) 0 (d) 1

39. $\lim_{x\to\pi/2}\frac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}=$

(a) $\ln a$ (b) $\ln 2$ (c) a (d) $\ln x$

40. If $f(x)=\begin{vmatrix}\sin x & \cos x & \tan x\\ x^3 & x^2 & x\\ 2x & 1 & 1\end{vmatrix}$, then $\lim_{x\to 0}\frac{f(x)}{x^2}$ is

(a) 3 (b) –1 (c) 0 (d) 1

41. If $f(x)=\cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ and $g(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then $\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$, $0<a<\frac{1}{2}$ is

(a) $\frac{3}{2(1+a^2)}$ (b) $\frac{3}{1+a^2}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

42. $\lim_{x\to 0}\frac{\tan x(1-\cos x)}{(1+\tan x)x^3}$ is equal to

(a) $\frac{1}{8}$ (b) 0 (c) $\frac{1}{2}$ (d) ∞

43. If $\lim_{x\to 0}\frac{(2(a-2)x-\tan x)\sin 2x}{x^2}=0$, then a is equal to

(a) 0 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

44. $\lim_{h\to 0}\frac{\log_e(1+4h)-2\log_e(1+2h)}{h^2}$

(a) – 4 (b) 4 (c) 2 (d) – 2

45. $\lim_{x\to a}\frac{\ln(1+(x-a))}{(x-a)}=$

(a) – 1 (b) 2 (c) 1 (d) – 2

46. $\lim_{h\to 0}\frac{\log_{10}(1+h)}{h}=$

(a) 1 (b) $\log_{10}e$ (c) $\log_e 10$ (d) None of these

47. If $\lim_{x\to 0}\dfrac{\log(3+2x)-\log(3-2x)}{x}=k$, then the value of k is

(a) 0 (b) $-\dfrac{2}{3}$ (c) $\dfrac{4}{3}$ (d) $-\dfrac{4}{3}$

48. $\lim_{x\to 0}\dfrac{e^{2x}-e^{x}-x}{x^2}=$

(a) 0 (b) -1 (c) $\dfrac{1}{2}$ (d) $\dfrac{3}{2}$

49. The value of $\lim_{x\to 0}\dfrac{e^{x}-1-x}{x^2}$ is

(a) 0 (b) $\dfrac{1}{2}$ (c) 1 (d) $\dfrac{1}{4}$

50. $\lim_{x\to 0}\dfrac{2^{x}-1}{\sqrt{1+x}-1}$ is equal to

(a) $\ln 4$ (b) $\ln\sqrt{2}$ (c) $\ln 2$ (d) $\ln\dfrac{1}{2}$

51. The value of $\lim_{x\to\infty}\left(\dfrac{x+4}{x+1}\right)^{x+3}$ is

(a) e^4 (b) e^3 (c) 1 (d) e^2

52. If a, b, c, d are positive, then $\lim_{x\to\infty}\left(1+\dfrac{1}{a+bx}\right)^{c+dx}$

(a) $e^{d/b}$ (b) $e^{c/a}$ (c) $e^{(c+d)/(a+b)}$ (d) e

53. $\lim_{x\to 0}(2x)^{3x}=$

(a) 0 (b) 1 (c) e (d) None of these

54. The value of $\lim_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-e+\dfrac{ex}{2}}{x^2}$ is

(a) $\dfrac{11e}{24}$ (b) $\dfrac{-11e}{24}$ (c) $\dfrac{e}{24}$ (d) None of these

55. $\lim_{m\to\infty}\left(\cos\dfrac{x}{m}\right)^{m}=$

(a) 0 (b) e (c) 1/e (d) 1

56. $\lim_{n\to\infty}\left(\dfrac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}=$

(a) e (b) e^2 (c) 1/e (d) 1

57. $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=$

(a) $\frac{m}{n}$ (b) $\frac{n}{m}$ (c) $\frac{m^2}{n^2}$ (d) $\frac{n^2}{m^2}$

58. $8\left(\lim_{x\to\frac{\pi}{8}}\frac{\sin 2x-\cos 2x}{8x-\pi}\right)=$

(a) $2\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) $\sqrt{2}$

59. $\lim_{x\to a}\frac{x^3-a^3}{\sin(x-a)}=$

(a) 0 (b) $\frac{3a^2}{2}$ (c) 3a (d) $3a^2$

60. $\lim_{h\to 0}\frac{\sqrt{4+\sin h}-2}{h}=$

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 0 (d) Does not exist

61. $\lim_{\alpha\to\beta}\frac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}$

(a) 0 (b) 1 (c) $\frac{\sin^2\beta}{\beta^2}$ (d) $\frac{\sin 2\beta}{2\beta}$

62. $\lim_{x\to 0}\frac{\tan 4x-2x}{6x-\sin 3x}$ equals

(a) 2/3 (b) 1/3 (c) 1/2 (d) 0

63. $\lim_{x\to 1}\frac{5-\sqrt{26-x^2}}{x-1}$ equals

(a) $\frac{1}{\sqrt{26}}$ (b) $\frac{1}{5}$ (c) $-\sqrt{24}$ (d) $\frac{1}{\sqrt{24}}$

64. If $f(1)=2, f'(1)=1, g(1)=1, g'(1)=2$, then $\lim_{x\to 1}\frac{g(x)f(1)-g(1)f(x)}{x-1}$ equals

(a) -3 (b) $\frac{1}{3}$ (c) 3 (d) $-\frac{1}{3}$

65. Value of $\lim_{x\to-\infty}\frac{\sqrt{x^2-1}+\sqrt{x^2-2}}{x+1}$ is

(a) -2 (b) $\frac{1}{2}$ (c) 2 (d) $-\frac{1}{2}$

EXERCISE - 1 (B)

1. If $f(x)=\begin{cases} x^2+2, x\geq 1 \\ 2x+1, x<1 \end{cases}$, then $\lim\limits_{x\to 1} f(x)$ equals

(a) 1 (b) 2 (c) 3 (d) Does not exist

2. $\lim\limits_{x\to 0} \dfrac{1+e^{-1/x}}{1-e^{-1/x}}$ is equal to

(a) 1 (b) –1 (c) 0 (d) Does not exist

3. If $f(x)=\begin{cases} x-1, x<0 \\ 1/4, x=0 \\ x^2, x>0 \end{cases}$ then $\lim\limits_{x\to 0} f(x)$ equals

(a) 0 (b) 1 (c) –1 (d) Does not exist

4. $\lim\limits_{x\to 3} \dfrac{x-3}{|x-3|}$, is equal to

(a) 1 (b) –1 (c) 0 (d) Does not exist

5. $\lim\limits_{x\to\infty} \dfrac{2x^2+3x}{3x^2+4}$ equals

(a) 1/2 (b) 2/3 (c) 3/4 (d) 0

6. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+1}-x\right)$ equals

(a) –1 (b) 0 (c) 1 (d) None of these

7. If $f(x)=\sqrt{\dfrac{x-\sin x}{x+\cos^2 x}}$, then $\lim\limits_{x\to\infty} f(x)$ equals

(a) 0 (b) ∞ (c) 1 (d) None of these

8. $\lim\limits_{x\to a} \left[\dfrac{x^2-(a+1)x+a}{x^3-a^3}\right]$ is equal to

(a) $\dfrac{a-1}{3a^2}$ (b) a – 1 (c) a (d) 0

9. $\lim\limits_{x\to 0} \dfrac{\tan x}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$ is equal to

(a) 1/2 (b) 2 (c) 1 (d) 0

10. $\lim_{x\to 0}\dfrac{\tan 2x - x}{3x - \sin x}$ equals

(a) 2/3 (b) 1/3 (c) 1/2 (d) 0

11. $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$ is equal to

(a) $e^{-1/3}$ (b) $e^{1/3}$ (c) $e^{1/6}$ (d) $e^{-1/6}$

12. $\lim_{x\to 0}\dfrac{\sin x^{o}}{x}$ is equal to

(a) 1 (b) π (c) x (d) $\pi/180$

13. $\lim_{x\to\infty}\left(\dfrac{\pi}{2}-\tan^{-1}x\right)^{1/x}$ equals

(a) 0 (b) 1 (c) ∞ (d) -1

14. If $G(x) = -\sqrt{25-x^2}$, then $\lim_{x\to 1}\dfrac{G(x)-G(1)}{x-1}$ equals

(a) 1/24 (b) 1/5 (c) $-\sqrt{24}$ (d) None of these

15. If f(9) = 9 and f'(9) = 4, then $\lim_{x\to 9}\dfrac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ is equal to -

(a) 1 (b) 3 (c) 4 (d) 9

16. Let $f(1) = g(1) = k$ and their n^{th} derivatives $f^{n}(1), g^{n}(1)$ exist and are not equal for some n.

If $\lim_{x\to 1}\dfrac{f(1)\,g(x)-f(1)-g(1)\,f(x)+g(1)}{g(x)-f(x)} = 4,$ then the value of k is

(a) 4 (b) 2 (c) 1 (d) 0

17. The value of $\lim_{x\to 0}\left(\dfrac{\int_0^{2x^2}\sec^2 2t\,dt}{x\sin x}\right)$ is

(a) 4 (b) 2 (c) 1 (d) 0

18. The value of $\lim_{x\to 0}\dfrac{\sin^{-1}\left(3x-4x^3\right)}{\ln\left(1+2x\right)}$ is

(a) 4 (b) 2 (c) 3/2 (d) 6

19. The value of $\lim_{x\to 2}\frac{(x^2+5)^{\frac{1}{2}}-(x^3+1)^{\frac{1}{2}}}{x^2-4}$ is

(a) 4/3 (b) 2/3 (c) –1/3 (d) 1/6

20. The value of $\lim_{x\to 0}\frac{\tan 2x-\sin x}{x}$ is

(a) 1 (b) 0 (c) 2 (d) 1/6

21. The value of $\lim_{x\to 0}\frac{\sin 4x-\sin 2x+\sin x}{x\cot x+1}$ is

(a) 1 (b) 0 (c) 2 (d) 2/3

22. The value of $\lim_{x\to 0}\frac{\ln(1+\sin x)}{e^{\tan 2x}-1}$ is

(a) 1 (b) 1/2 (c) 2 (d) 0

23. The value of $\lim_{x\to \frac{\pi}{4}}(1+\cos 2x)^{\frac{1}{1-\tan x}}$ is

(a) e (b) 1/e (c) e^2 (d) $1/e^2$

24. If $\lim_{x\to 0}\frac{a\sin x+b\cos x+ce^x}{x^2}=2$, then

(a) $a=b=2, c=-2$
(b) $a=b=c=-2$
(c) $a=b=c=2$
(d) $a=b=-2, c=2$

25. If $\lim_{x\to\infty}\left(\frac{ax+1}{ax+2}\right)^{2x}=e^{1/2}$, then $a=$

(a) 1 (b) 2 (c) –4 (d) 1/2

26. The limit $\lim_{x\to 0}\frac{\sin[x]}{[x]}$, where [x] denotes greatest integer less than or equal to x, is equal to

(a) 1 (b) 0 (c) sin1 (d) not defined

27. The limit $\lim_{x\to 0}(1+[x])^{\frac{1}{x}}$, where [x] denotes greatest integer less than or equal to x, is equal to

(a) 1 (b) e (c) 0 (d) does not exixt

28. $\lim_{x\to\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right)=$

(a) 0 (b) ∞ (c) 2 (d) 1/2

29. $\lim_{x\to 0}\frac{2x\left(3^x-1\right)}{1-\cos 2x}$ is equal to

(a) $\ln 3$ (b) $\ln 9$ (c) 0 (d) $\ln\sqrt{3}$

30. The value of $\lim_{n\to\infty}\frac{1^5+2^5+3^5+........+n^5}{n^6}=$

(a) ∞ (b) 1/2 (c) 0 (d) 1/6

31. The value of $\lim_{n\to\infty}\left[\frac{3}{1+n^3}+\frac{12}{8+n^3}+\frac{27}{27+n^3}+....+n\text{ terms}\right]$ is equal to

(a) $\ln 3$ (b) $\ln 2$ (c) 0 (d) 1

32. The value of $\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{\sin^{-1}2x}=$

(a) 2 (b) 1/2 (c) 0 (d) 1/4

33. The value of $\lim_{x\to\infty}\left(\frac{1^x+3^x+5^x+...+(2n-1)^x}{n}\right)^{\frac{1}{x}}$ is

(a) $2n-1$ (b) $(2n-1)/n$ (c) 0 (d) 1

34. The value of $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+...n^x}{n}\right)^{\frac{1}{x}}$ is

(a) 1 (b) $(n!)^{1/n}$ (c) 0 (d) $(n!)^n$

35. $\lim_{x\to 0}\frac{8\left(2^x-3^x\right)\tan x}{1-\cos 4x}=$

(a) ∞ (b) $\ln\frac{3}{2}$ (c) 0 (d) $\ln\frac{2}{3}$

EXERCISE - 1(C)

1. The value of $\lim_{x\to 1}\dfrac{\sqrt[13]{x}-\sqrt[7]{x}}{\sqrt[5]{x}-\sqrt[3]{x}}$ is

(a) $\dfrac{9}{13}$ (b) $\dfrac{5}{91}$ (c) $\dfrac{9}{91}$ (d) $\dfrac{45}{91}$

2. The value of $\lim_{x\to 1}\dfrac{\left[\sum_{K=1}^{100}x^k\right]-100}{x-1}$ is

(a) 4950 (b) 5050 (c) 5150 (d) 5151

3. The value of $\lim_{x\to 1}\left(\dfrac{p}{1-x^p}-\dfrac{q}{1-x^q}\right)p, q\in N$, is

(a) $\dfrac{p-q}{2}$ (b) $\dfrac{p+q}{2}$ (c) $\dfrac{q-p}{2}$ (d) None of these

4. The value of $\lim_{x\to \frac{3\pi}{4}}\dfrac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x}$ is

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{3}$ (c) $-\dfrac{1}{2}$ (d) $-\dfrac{1}{3}$

5. The value of $\lim_{x\to 0}\left(\ln(1+\sin^2 x)\right)\cot\left(\ln^2(1+x)\right)$ is

(a) 0 (b) 1 (c) -1 (d) $\dfrac{1}{2}$

6. The value of $\lim_{x\to\infty}\dfrac{2\sqrt{x}+3x^{1/3}+5x^{1/5}}{\sqrt{3x-2}+(2x-3)^{1/3}}$ is

(a) 1 (b) $\dfrac{1}{\sqrt{3}}$ (c) $\dfrac{2}{\sqrt{3}}$ (d) $\sqrt{3}$

7. The value of $\lim_{x\to 0}\dfrac{\sec 4x-\sec 2x}{\sec 3x-\sec x}$ is

(a) 1 (b) $\dfrac{2}{3}$ (c) $\dfrac{3}{2}$ (d) 2

8. The sum of an infinite geometric series whose first term is the limit of the function $f(x)=\dfrac{\tan x-\sin x}{\sin^3 x}$ as $x\to 0$ and whose common ratio is the limit of the function $g(x)=\dfrac{1-\sqrt{x}}{(\cos^{-1}x)^2}$ as $x\to 1$, is

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{2}{3}$

9. The value of $\lim_{x\to\infty}\left(x-\ln\left(\frac{e^x+e^{-x}}{2}\right)\right)$ is

(a) 0 (b) $\ln 2$ (c) 1 (d) $2\ln 2$

10. The value of $\lim_{x\to 0}\frac{8}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$ is

(a) $\frac{1}{16}$ (b) $\frac{1}{32}$ (c) $\frac{1}{64}$ (d) $\frac{1}{8}$

11. The value of $\operatorname{Lim}_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$ is

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{4\sqrt{2}}$ (c) $\frac{1}{8\sqrt{2}}$ (d) $\frac{1}{16\sqrt{2}}$

12. The value of $\operatorname{Lim}_{x\to\frac{\pi}{2}}\frac{2^{-\cos x}-1}{x(x-\frac{\pi}{2})}$ is

(a) $\frac{\ln 2}{\pi}$ (b) $\frac{2\ln 2}{\pi}$ (c) $\frac{\ln 8}{\pi}$ (d) $\frac{\ln 16}{\pi}$

13. The value of $\operatorname{Lim}_{x\to 1}\frac{(ln(1+x)-ln\,2)(3.4^{x-1}-3x)}{[(7+x)^{\frac{1}{3}}-(1+3x)^{\frac{1}{2}}].\sin(x-1)}$ is

(a) $\frac{3}{4}\ln\frac{4}{e}$ (b) $-\frac{3}{4}\ln\frac{4}{e}$ (c) $\frac{9}{4}\ln\frac{4}{e}$ (d) $-\frac{9}{4}\ln\frac{4}{e}$

14. The value of $\operatorname{Lim}_{n\to\infty}\sum_{r=2}^{n}\left((r+1)\sin\frac{\pi}{r+1}-r\sin\frac{\pi}{r}\right)$ is

(a) $\pi-2$ (b) $\frac{\pi}{2}-1$ (c) $2\pi-2$ (d) $\frac{\pi}{4}-\frac{1}{2}$

15. The value of $\operatorname{Lim}_{x\to-\infty}\frac{(3x^4+2x^2)\sin\frac{1}{x}+|x|^3+5}{|x|^3+|x|^2+|x|+1}$ is

(a) 2 (b) -2 (c) 1 (d) -1

16. The value of $\operatorname{Lim}_{x\to 3}\frac{(x^3+27)\ln(x-2)}{x^2-9}$ is

(a) 1 (b) 3 (c) 6 (d) 9

17. The value of $\operatorname{Lim}_{x\to 0}\frac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}$ is

(a) $8\sqrt{2}(\ln 3)^2$ (b) $4\sqrt{2}(\ln 3)^2$ (c) $2\sqrt{2}(\ln 3)^2$ (d) $\sqrt{2}(\ln 3)^2$

18. The value of $\lim_{x\to 0}\left[\frac{(1+x)^{1/x}}{e}\right]^{1/x}$ is

(a) $\sqrt{e}$ (b) $\frac{1}{\sqrt{e}}$ (c) $\frac{1}{e}$ (d) e

19. The value of $\lim_{n\to\infty}\left(\frac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$ is

(a) $\sqrt{e}$ (b) $\frac{1}{\sqrt{e}}$ (c) $\frac{1}{e}$ (d) e

20. The value of $\lim_{x\to 1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$ is

(a) e (b) e^{-1} (c) $\sqrt{e}$ (d) $\frac{1}{\sqrt{e}}$

21. $\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{1/x}$ is equal to

(a) e^{-1} (b) e (c) e^2 (d) $\sqrt{e}$

22. $\lim_{x\to 0}\left(\frac{3^x-1}{x}\right)$ equals

(a) log 3 (b) 3 log 3 (c) 2 log 3 (d) None of these

23. $\lim_{x\to 0}\frac{\cos(\sin x)-1}{x^2}=$

(a) 1 (b) – 1 (c) 1/2 (d) – 1/2

24. $\lim_{x\to 0}\frac{e^{1/x}}{e^{\frac{1}{x}}+1}=$

(a) 0 (b) 1 (c) 2 (d) Does not exist

25. $\lim_{x\to 0}\frac{a^{\sin x}-1}{b^{\sin x}-1}=$

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$ (c) $\frac{\log a}{\log b}$ (d) $\frac{\log b}{\log a}$

26. $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=$

(a) $\frac{1}{2}$ (b) ∞ (c) 1 (d) 0

27. $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+1} =$

(a) e^2 (b) e^3 (c) e (d) e^{-1}

28. $\lim_{x\to 0}(1-ax)^{\frac{1}{x}} =$

(a) e (b) e^{-a} (c) 1 (d) e^{a}

29. If $\lim_{x\to 0}\frac{x^n-\sin x^n}{x-\sin^n x}$ is non zero definite, then n must be

(a) 1 (b) 2 (c) 3 (d) None of these

30. The values of a and b such that $\lim_{x\to 0}\frac{x(2+a\cos x)+b\sin x}{x^3}=2,$ are

(a) $a=5, b=-3$ (b) $a=-5, b=3$ (c) $a=-5, b=-3$ (d) $a=5, b=3$

31. $\lim_{h\to 0}\frac{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)}{\sqrt{3}h(\sqrt{3}\cos h-\sin h)} =$

(a) $-\frac{1}{3}$ (b) $-\frac{3}{8}$ (c) $-\sqrt{3}$ (d) $\frac{2}{3}$

32. The value of $\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}$ is

(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) None of these

33. $\lim_{x\to\frac{\pi}{4}}\left(\frac{1-\tan x}{1-\sqrt{2}\sin x}\right)$ equals

(a) 0 (b) 1 (c) -2 (d) 2

34. $\lim_{x\to 0}\frac{3^x-1}{\sqrt{4+x}-2}$ equals

(a) $\log 9$ (b) $\log 81$ (c) $\log 3$ (d) $\log 27$

35. $\lim_{x\to 0}\left(\frac{\sin(2+2x)+\sin(2-2x)-2\sin 2}{x\sin x}\right)$ is equal to

(a) $4\sin 2$ (b) $-4\sin 2$ (c) 1 (d) $-2\sin 2$

36. $\lim_{x\to\frac{\pi}{4}}\frac{\int_{\pi/4}^{x} t^2\,dt}{\cos 2x}$ is equal to

(a) ∞ (b) $\frac{\pi^2}{32}$ (c) $-\frac{\pi^2}{32}$ (d) $-\frac{\pi^2}{16}$

37. $\lim_{h\to 0}\frac{(x+h)\sec(x+h)-x\sec x}{\sin h}=$

(a) $\sec x(x\tan x+1)$ (b) $x\tan x+\sec x$

(c) $x\sec x+\tan x$ (d) None of these

38. $\lim_{x\to \frac{\pi}{6}}\frac{\sin\left(\frac{\pi}{3}-2x\right)}{2\cos 2x-1}$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\frac{2}{\sqrt{3}}$

39. $\lim_{x\to 0}\frac{e^{x^4}-\cos^2 x}{x^4}$ is equal to

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) 3

40. $\lim_{x\to 0}\frac{x\cos^2 x-\sin^2 x}{x+\sin x}$ equals

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) -1

WINDOW TO JEE MAIN

1. $\lim_{x\to 0}\frac{\ln x^n-[x]}{[x]}, n\in N$, ([x] denotes greatest integer less than or equal to x)

(a) has value – 1 (b) has value 0 (c) has value 1 (d) does not exist

(2002)

2. If $f(1)=1, f'(1)=2$, then $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ is

(a) 2 (b) 4 (c) 1 (d) 1/2

(2002)

3. $\lim_{x\to 0}\frac{\sqrt{1-\cos 2x}}{\sqrt{2}\,x}$ is

(a) 1 (b) –1 (c) 0 (d) does not exist.

(2002)

4. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^{\frac{1}{x}}=$

(a) e^4 (b) e^2 (c) e^3 (d) 1

(2002)

5. Let $f(2)=4$ and $f'(2)=4$ then $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$ equals

(a) 2 (b) – 2 (c) – 4 (d) 3

(2002)

6. If $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}=k$ the value of k is

(a) –1/3 (b) 2/3 (c) –2/3 (d) 0

(2003)

7. The value of $\lim_{n\to\infty}\frac{1+2^4+3^4+...+n^4}{n^5}-\lim_{n\to\infty}\frac{1+2^3+3^3+...+n^3}{n^5}$ is

(a) zero (b) 1/4 (c) 1/5 (d) 1/30

(2003)

8. $\lim_{x\to\frac{\pi}{2}}\frac{\left(1-\tan\frac{x}{2}\right)(1-\sin x)}{\left(1+\tan\frac{x}{2}\right)(\pi-2x)^3}$ is

(a) 0 (b) 1/32 (c) ∞ (d) 1/8

(2003)

9. Let $f(a)=g(a)=k$ and their n^{th} derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n.

Further if $\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$, then the value of k is

(a) 2 (b) 1 (c) 0 (d) 4

(2003)

10. If $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$, then the values of a and b, are

(a) $a \in \mathbf{R}, b = 2$ (b) $a = 1, b \in \mathbf{R}$
(c) $a \in \mathbf{R}, b \in \mathbf{R}$ (d) $a = 1$ and $b = 2$

(2004)

11. Let a and b be the distinct roots of $ax^2 + bx + c = 0$, then $\lim_{x\to\alpha}\frac{1-\cos\left(ax^2+bx+c\right)}{\left(x-\alpha\right)^2}$ is equal to

(a) 0 (b) $\frac{a^2\left(\alpha-\beta\right)^2}{2}$ (c) $\frac{\left(\alpha-\beta\right)^2}{2}$ (d) $-\frac{a^2\left(\alpha-\beta\right)^2}{2}$

(2005)

12. Let $f: R \to R$ be a positive increasing function with $\lim_{x\to\infty}\frac{f(3x)}{f(x)}=1$, then $\lim_{x\to\infty}\frac{f(2x)}{f(x)}=$

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 3

(2010)

13. Let p(x) be a function defined on R such that, $p'(x) = p'(1-x)$, for all $x \in [0,1]$, $p(0) = 1$ and $p(1) = 41$.

Then $\int_0^1 p(x)\,dx$ equals

(a) 41 (b) 21 (c) 41 (d) 42

(2010)

14. $\lim_{x\to 2}\frac{\sqrt{1-\cos 2(x-2)}}{x-2}$

(a) equals $\frac{1}{\sqrt{2}}$ (b) equals $\sqrt{2}$ (c) equals $-\sqrt{2}$ (d) does not exist

(2011)

15. $\lim_{x\to 0}\frac{(1-\cos 2x)(\cos x+3)}{x\tan 4x}=$

(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

(2013)

EXERCISE - 2 (A)

More than one options may be correct

1. Assume that $\lim_{\theta\to-1} f(\theta)$ exists and $\frac{\theta^2+\theta-2}{\theta+3} \le \frac{f(\theta)}{\theta^2} \le \frac{\theta^2+2\theta-1}{\theta+3}$ holds for certain interval containing the point $\theta = -1$ then $\lim_{\theta\to-1} f(\theta)$

(A) is equal to $f(-1)$ (B) is equal to 1 (C) is non existent (D) is equal to – 1

2. Let $l_1 = \lim_{x\to\infty}\sqrt{\frac{x-\cos^2 x}{x+\sin x}}$ and $l_2 = \lim_{h\to0^+}\int_{-1}^{1}\frac{h\,dx}{h^2+x^2}$. Then

(A) both l_1 and l_2 are less than $\frac{22}{7}$

(B) one of the two limits is rational and other irrational.

(C) $l_2 > l_1$

(D) l_2 is greater than 3 times of l_1.

3. $\lim_{x\to c} f(x)$ does not exist when

(A) $f(x) = [[x]] - [2x-1]$, c = 3 (B) $f(x) = [x] - x$, c = 1

(C) $f(x) = \{x\}^2 - \{-x\}^2$, c = 0 (D) $f(x) = \frac{\tan(\text{sgn}\,x)}{\text{sgn}\,x}$, c = 0 .

where [x] denotes step up function & {x} fractional part function.

4. Let $f(x) = \begin{cases} \frac{\tan^2\{x\}}{x^2-[x]^2} & \text{for } x>0 \\ 1 & \text{for } x=0 \\ \sqrt{\{x\}\cot\{x\}} & \text{for } x<0 \end{cases}$ where [x] is the step up function and { x } is the fractional part function of x, then :

(A) $\lim_{x\to0^+} f(x) = 1$ (B) $\lim_{x\to0^-} f(x) = 1$

(C) $\cot^{-1}\left(\lim_{x\to0^-} f(x)\right)^2 = 1$ (D) f is continuous at x = 1.

5. Which of the following limits vanish?

(A) $\text{Limit}_{x\to\infty}\ x^{\frac{1}{4}}\sin\frac{1}{\sqrt{x}}$ (B) $\text{Limit}_{x\to\pi/2}\ (1-\sin x).\tan x$

(C) $\text{Limit}_{x\to\infty}\ \frac{2x^2+3}{x^2+x-5}.\text{sgn}(x)$ (D) $\text{Limit}_{x\to3^+}\ \frac{[x]^2-9}{x^2-9}$

where [ ] denotes greatest integer function

6. Let $f(x)=\lim_{n\to\infty}\dfrac{2x^{2n}\sin\frac{1}{x}+x}{1+x^{2n}}$ then which of the following alternative(s) is/are correct?

(A) $\lim_{x\to\infty} x f(x)=2$

(B) $\lim_{x\to 1} f(x)$ does not exist.

(C) $\lim_{x\to 0} f(x)$ does not exist.

(D) $\lim_{x\to -\infty} f(x)$ is equal to zero.

7. If $\lim_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2}-\sqrt{x^4+2x^3-cx^2+3x-d}\right)=4$, then

(A) a = 2 (B) b ∈ R (C) c = 5 (D) d ∈ R

8. If $\lim_{x\to 0}\dfrac{a+b\sin x-\cos x+ce^x}{x^3}$ exists finitely, then

(A) a = 2 (B) b = 1 (C) c = 1 (D) d ∈ R

9. If $f(x)=\begin{cases}\dfrac{a^{[x]+x}}{[x]+x}, & x\neq 0\\ \ln a, & x=0\end{cases}$, then which of the following is correct

(A) $\lim_{x\to 0^-} f(x)=1-\dfrac{1}{a}$

(B) $\lim_{x\to 0^+} f(x)=\ln a$

(C) $\lim_{x\to 0} f(x)$ does not exist

(D) f(x) is continuous at x = 0

10. Let $f(x)=\left(\dfrac{x}{2+x}\right)^{2x}$, then

(A) $\underset{x\to\infty}{\text{Lt}}\, f(x)=e^{-4}$ (B) $\underset{x\to 1}{\text{Lt}}\, f(x)=\dfrac{1}{9}$ (C) $\underset{x\to\infty}{\text{Lt}}\, f(x)=e^{4}$ (D) none of these

11. If $\lim_{x\to 0}\dfrac{ae^{1/x}+be^{-1/x}}{ce^{1/x}+de^{-1/x}}=2$, then the roots of $bx^2+(a-2c)x-2d=0, a,b,c,d\in Q$ are

(A) real (B) integral (C) rational (D) equal

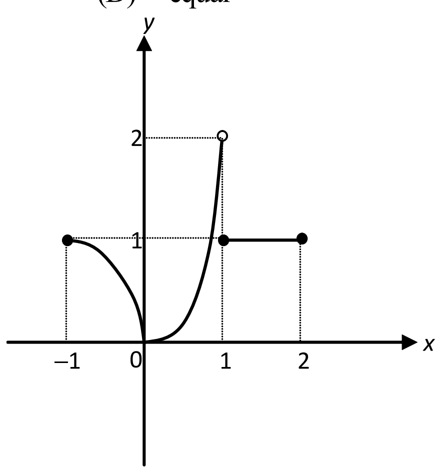
12. Which of the following statements are true of the function f defined for $-1\le x\le 3$ in the figure shown.

(A) $\lim_{x\to -1^+} = 2$

(B) $\lim_{x\to 2} f(x)$ does not exist

(C) $\lim_{x\to 1^-} f(x)=2$

(D) $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)$



13. For all positive real valued functions f & g defined for real values of x. Let R be a relation defined by the statement that "f is related to g only & only if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$". Which of the following can be concluded if f is related to g?

(A) f^2 is related to g^2
(B) $\sqrt{f}$ is related to $\sqrt{g}$
(C) e^f is related to e^g
(D) f + g is related to 2g

14. If $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}=e^{-\frac{2}{\pi}}$, then which of the following is correct

(A) $a\in(-\infty,0)$ (B) $a\in R$ (C) $a\in(0,\infty)$ (D) $a\in(-1,1)$

15. Which of the following is incorrect? ([x] denotes greatest integer less than or equal to x)

(A) $\left[\lim_{x\to\infty}\frac{2x^2+1}{x^2+2}\right]=2$ (B) $\left[\lim_{x\to 0}\frac{\sin 2x}{x}\right]=1$ (C) $\left[\lim_{x\to 0}\frac{\tan 3x}{x}\right]=3$ (D) $\left[\lim_{x\to -\infty}\frac{\sqrt{4x^2+1}}{x+2}\right]=-2$

PASSAGE 1

Let f(x) is a function continuous for all $x\in R$ except at $x=0$. Such that $f'(x)<0\ \forall\ x\in(-\infty,0)$ and $f'(x)>0\ \forall\ x\in(0,\infty)$. Let $\lim_{x\to 0^+}f(x)=2$, $\lim_{x\to 0^-}f(x)=3$ and $f(0)=4$.

16. The value of λ for which $2\left(\lim_{x\to 0}f(x^3-x^2)\right)=\lambda\left(\lim_{x\to 0}f(2x^4-x^5)\right)$ is

(A) $\frac{4}{3}$ (B) 2 (C) 3 (D) 5

17. The values of $\lim_{x\to 0^+}\frac{f(-x)x^2}{\left\{\frac{1-\cos x}{[f(x)]}\right\}}$ where [·] denote greatest integer function and { · } denote fraction part function.

(A) 6 (B) 12 (C) 18 (D) 24

18. $\lim_{x\to 0^-}\left(\left[3f\left(\frac{x^3-\sin^3 x}{x^4}\right)\right]-f\left(\left[\frac{\sin x^3}{x}\right]\right)\right)$ where [·] denote greatest integer function.

(A) 3 (B) 5 (C) 7 (D) 9

PASSAGE 2

Suppose that function $f: R\to R$ satisfies the inequality, $\left|\sum_{k=1}^{n}\left(3^k\left\{f(x+ky)-f(x-ky)\right\}\right)\right|\le 1$ for every positive integer n and for all $x,y\in R$. Also let $g(x)=ax^3-(3+b)x^2+cx$ for $x\in R$, where a, b, c are are such that $\lim_{x\to 0}\frac{axe^x-b\ln(1+x)+cxe^{-x}}{x^2\sin x}=2$.

19. If $f(13)=17$, then the value of $f(293)$ is
(A) 17 (B) 297 (C) 13 (D) 293

20. If $f(13) = -23$, number of solutions of $g(x)=f(x)$ is
(A) 2 (B) 3 (C) 4 (D) 1

21. Value of $\lim_{x\to 1}\frac{g(x)+3}{\ln(2-x)}$ is equal to
(A) 12 (B) 3 (C) 9 (D) None of these

PASSAGE 3

Let $f(x)=\lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}\right)^n, g(x)\lim_{n\to\infty}\left(1+x\left(1+e^{1/n}\right)\right)^n$ and $h(x)=\tan^{-1}\left(g^{-1}\left(f^{-1}(x)\right)\right)$.

22. Value of $\lim_{x\to 0}\frac{\ln(f(x))}{\ln(g(x))}$ is equal to
(A) 1/2 (B) -1/2 (C) 0 (D) 1

23. Domain of the function h(x) is
(A) $(0,\infty)$ (B) R (C) (0, 1) (D) [0, 1]

24. Range of the function h(x) is
(A) $\left(0,\frac{\pi}{2}\right)$ (B) $\left(-\frac{\pi}{2},0\right)$ (C) R (D) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

MATRIX MATCH TYPE

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	Column-I		Column-II
(A)	$\lim_{x\to 0}\frac{1-\cos 2x}{e^{x^2}-e^x+x}$ equals	(P)	1
(B)	If the value of $\lim_{x\to 0^+}\left(\frac{(3/x)+1}{(3/x)-1}\right)^{1/x}$ can be expressed in the form of $e^{p/q}$, where p and q are relative prime then $(p+q)$ is equal to	(Q)	2
(C)	$\lim_{x\to 0}\frac{\tan^3 x-\tan x^3}{x^5}$ equals	(R)	4
(D)	$\lim_{x\to 0}\frac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$	(S)	5

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	Column-I		Column-II
(A)	$\lim_{n\to\infty}\cos^2\left(\pi\left(\sqrt[3]{n^3+n^2+2n}\right)\right)$ where n is an integer, equals	(P)	$\frac{1}{2}$
(B)	$\lim_{n\to\infty} n\sin\left(2\pi\sqrt{1+n^2}\right)$ $(n\in N)$ equals	(Q)	$\frac{1}{4}$
(C)	$\lim_{n\to\infty}(-1)^n\sin\left(\pi\sqrt{n^2+0.5n+1}\right)\left(\sin\frac{(n+1)\pi}{4n}\right)$ is (where $n\in N$)	(R)	π
(D)	If $\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x = e$ where 'a' is some real constant then the value of 'a' is equal to	(S)	non existent

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	Column-I		Column-II
(A)	$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right)$ equals	(P)	-2
(B)	The value of the limit, $\lim_{x\to 0}\dfrac{\sin 2x-2\tan x}{ln(1+x^3)}$ is	(Q)	-1
(C)	$\lim_{x\to 0^+}\left(ln\sin^3 x-ln(x^4+ex^3)\right)$ equals	(R)	0
(D)	Let $\tan(2\pi\,|\sin\theta|)=\cot(2\pi\,|\cos\theta|)$, where $\theta\in R$ and $f(x)=(|\sin\theta|+|\cos\theta|)^x$. The value of $\lim_{x\to\infty}\left[\dfrac{2}{f(x)}\right]$ equals (Here [ ] represents greatest integer function)	(S)	1

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	Column–I		Column-II
(A)	$\lim_{x\to 1}\dfrac{ln\,x}{x^4-1}$	(P)	2
(B)	$\lim_{x\to 0}\dfrac{3e^x-x^3-3x-3}{\tan^2 x}$	(Q)	$\dfrac{2}{3}$
(C)	$\lim_{x\to\infty}\dfrac{\pi-2\tan^{-1}x}{ln\left(1+\dfrac{1}{x}\right)}$	(R)	$\dfrac{3}{2}$
(D)	$\lim_{x\to 0}\dfrac{2\sin x-\sin 2x}{x(\cos x-\cos 2x)}$	(S)	$\dfrac{1}{4}$
(E)	$\lim_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$		

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	Column I		Column II
(A)	$\lim_{x\to\infty}\left(\dfrac{x}{1+x}\right)^x$ equals	(P)	e^2
(B)	$\lim_{x\to\infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^x$	(Q)	$e^{-1/2}$
(C)	$\lim_{x\to 0}(\cos x)^{\cot^2 x}$	(R)	e
(D)	$\lim_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{1/x}$	(S)	e^{-1}

EXERCISE - 2 (B)

1. If $\lim\limits_{x\to 0}\dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5} = C$, then find the value of A + B + C.

2. Find the value of $\lim\limits_{x\to 0}\dfrac{x^{6000}-(\sin x)^{6000}}{100\cdot x^2\cdot(\sin x)^{6000}}$.

3. If $\operatorname{Lim}\limits_{x\to 0}\dfrac{x+ln\left(\sqrt{x^2+1}-x\right)}{x^3}=\dfrac{p}{q}$ where p and q are coprimes then find the value of $\left|p^2-q^2\right|$.

4. At the end-points and the midpoint of a circular arc AB tangent lines are drawn, and the points A and B are joined with a chord. Find the limit of ratio of the areas of the two triangles thus formed as the arc AB decreases indefinitely.

5. If $\operatorname{Lim}\limits_{x\to 0}\dfrac{ae^x - b\cos x + ce^{-x}}{x.\sin x} = 2$, then find the value of a + b + c.

6. Let $L=\prod\limits_{n=3}^{\infty}\left(1-\dfrac{4}{n^2}\right)$; $M=\prod\limits_{n=2}^{\infty}\left(\dfrac{n^3-1}{n^3+1}\right)$ and $N=\prod\limits_{n=1}^{\infty}\dfrac{(1+n^{-1})^2}{1+2n^{-1}}$, then find the value of $L^{-1}+M^{-1}+N^{-1}$.

7. Find the value of $\operatorname{Lim}\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+..........+\dfrac{1}{\sqrt{n^2+2n}}\right)$.

8. Find the value of $\operatorname{Lim}\limits_{x\to 0^+} x^{\left(x^x-1\right)}$.

9. If $\operatorname{Lim}\limits_{x\to 0}\dfrac{1-\cos 3x\cdot\cos 9x\cdot\cos 27x.........\cos 3^n x}{1-\cos\frac{1}{3}x\cdot\cos\frac{1}{9}x\cdot\cos\frac{1}{27}x.........\cos\frac{1}{3^n}x} = 3^{10}$, find the value of n.

10. If the $\operatorname{Lim}\limits_{x\to 0}\dfrac{1}{x^3}\left(\dfrac{1}{\sqrt{1+x}}-\dfrac{1+ax}{1+bx}\right)$ exists and has the value equal to l, then find the value of $\dfrac{1}{8}\left(\dfrac{1}{a}-\dfrac{2}{l}+\dfrac{3}{b}\right)$.

11. If the equations , $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ & $x^2 + (a+b)x + 36 = 0$ have a common positive root, and the other three roots are also the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$, then evaluate $\lim\limits_{x\to 5}\dfrac{-P(x)}{\ln(x-4)^5}$.

12. Evaluate $\underset{x \to \pi/3}{\text{Limit}} \dfrac{3\tan x - \tan^3 x}{3\cos\left(x + \frac{\pi}{6}\right)}$.

13. Evaluate $\underset{x \to 1}{\text{Limit}} \left[\left[\dfrac{4}{x^2 - x^{-1}} - \dfrac{1 - 3x + x^2}{1 - x^3}\right]^{-1} + \dfrac{3 . \left(x^4 - 1\right)}{x^3 - x^{-1}}\right]$.

14. If $\lim\limits_{x\to 0} \dfrac{x\left(1 - m\cos x\right) + n\sin x}{x^3} = 1$, then find $m + n$.

15. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x}{x + \dfrac{x^{1/3}}{x + \dfrac{x^{1/3}}{x + \ldots\infty \text{ terms}}}}\right)$.

EXERCISE - 3

Subjecitve type

1. Let $f(x)=\frac{x}{\sin x},\ x>0$ and $g(x)=x+3,\quad x<1$
$=2-x,\ x\le 0$ $\quad =x^2-2x-2,\ 1\le x<2$
$=x-5,\quad x\ge 2$
find LHL and RHL of $g(f(x))$ at $x=0$ and hence find $\lim_{x\to 0} g(f(x))$.

2. Let $P_n = a^{P_{n-1}}-1$, $\forall\ n=2, 3,$.......and Let $P_1 = a^x - 1$ where $a \in R^+$ then evaluate $\lim_{x\to 0}\frac{P_n}{x}$.

3. Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences such that
(i) $a_n + b_n + c_n = 2n + 1$; **(ii)** $a_nb_n + b_nc_n + c_na_n = 2n - 1$; **(iii)** $a_nb_nc_n = -1$; **(iv)** $a_n < b_n < c_n$
Then find the value of $\lim_{n\to\infty} na_n$.

4. If $n \in N$ and $a_n = 2^2 + 4^2 + 6^2 + \ldots\ldots. + (2n)^2$ and $b_n = 1^2 + 3^2 + 5^2 + \ldots.. + (2n-1)^2$. Find the value $\lim_{n\to\infty}\frac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{n}}$.

5. At the end points A, B of the fixed segment of length L, lines are drawn meeting in C and making angles θ and 2θ respectively with the given segment. Let D be the foot of the altitude CD and let *x* represents the length of AD. Find the value of *x* as θ tends to zero i.e. $\lim_{\theta\to 0} x$.

6. $\lim_{x\to\infty}\left[\cos\left(2\pi\left(\frac{x}{1+x}\right)^a\right)\right]^{x^2}$ $\quad a \in R$

7. $\lim_{x\to 0}\left(\frac{x-1+\cos x}{x}\right)^{\frac{1}{x}}$

8. $\lim_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+a_3^{\frac{1}{x}}+\ldots..+a_n^{\frac{1}{x}}}{n}\right)^{nx}$ where $a_1,a_2,a_3,\ldots\ldots a_n > 0$

9. Let $f(x)=\frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\ .\ (1-\{x\})}$ then find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$, where $\{x\}$ denotes the fractional part function.

10. $\lim_{x\to a}\frac{1}{(a^2-x^2)^2}\left(\frac{a^2+x^2}{ax}-2\sin\left(\frac{a\pi}{2}\right)\sin\left(\frac{\pi x}{2}\right)\right)$ where a is an odd integer

11. If $L=\lim_{x\to 1}\frac{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^{2n})}{[(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^n)]^2}$ then show that L can be equal to
(a) $\prod_{r=1}^{n}\frac{n+r}{r}$ (b) $\frac{1}{n!}\prod_{r=1}^{n}(4r-2)$
(c) The sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$.
(d) The coefficient of x^n in the expansion of $(1+x)^{2n}$.

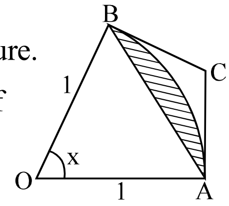
12. Evaluate, $\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x}$

13. $\lim_{y\to 0}\left[\text{Limit}_{x\to\infty}\frac{\exp\left(x\ln(1+\frac{ay}{x})\right)-\exp\left(x\ln(1+\frac{by}{x})\right)}{y}\right]$

14. Let $x_0 = 2\cos\frac{\pi}{6}$ and $x_n = \sqrt{2+x_{n-1}}$, n = 1, 2, 3, ..., find $\lim_{n\to\infty} 2^{(n+1)}\cdot\sqrt{2-x_n}$.

15. $\lim_{x\to 0}\left[\frac{\ln(1+x)^{1+x}}{x^2}-\frac{1}{x}\right]$

16. A circular arc of radius 1 subtends an angle of x radians, $0 < x < \frac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute:



(a) T(x) (b) S(x) & (c) the limit of $\frac{T(x)}{S(x)}$ as $x \to 0$.

17. Let $f(x) = \lim_{n\to\infty}\sum_{n=1}^{n} 3^{n-1}\sin^3\frac{x}{3^n}$ and $g(x) = x - 4f(x)$. Evaluate $\lim_{x\to 0}\left(1+g(x)\right)^{\cot x}$.

18. If $f(n,\theta) = \prod_{r=1}^{n}\left(1-\tan^2\frac{\theta}{2^r}\right)$, then compute $\lim_{n\to\infty} f(n,\theta)$

19. $L = \lim_{x\to 0}\frac{\sqrt{\frac{\cos 2x+(1+3x)^{1/3}}{2}}-\sqrt[3]{\frac{4\cos^3 x-\ln(1+x)^4}{4}}}{x}$

If $L = a/b$ where 'a' and 'b' are relatively primes find (a + b).

20. $\lim_{x\to\infty}\left(\frac{\cosh(\pi/x)}{\cos(\pi/x)}\right)^{x^2}$ where $\cosh t = \frac{e^t+e^{-t}}{2}$

21. Through a point A on a circle, a chord AP is drawn & on the tangent at A a point T is taken such that AT = AP. If TP produced meet the diameter through A at Q, prove that the limiting value of AQ when P moves upto A is double the diameter of the circle.

22. If $L = \lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{\ln(x+\sqrt{1+x^2})}\right)$ then find the value of $\frac{L+153}{L}$.

23. If $\lim_{x\to 0}\frac{1-\cos x\cdot\cos 2x\cdot\cos 3x........\cos nx}{x^2}$ has the value equal to 253, find the value of n (where $n \in N$).

24. Let $a_1 > a_2 > a_3$ .... $a_n > 1$; $p_1 > p_2 > p_3 .... > p_n > 0$; such that $p_1 + p_2 + p_3 + ... + p_n = 1$

Also $F(x) = \left(p_1a_1^x + p_2a_2^x + ... + p_na_n^x\right)^{1/x}$. Compute

(a) $\lim_{x\to 0^+} F(x)$ **(b)** $\lim_{x\to\infty} F(x)$ **(c)** $\lim_{x\to -\infty} F(x)$

25. Evaluate $\lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(r^2+\frac{3}{4}\right)$.

WINDOW TO JEE ADVANCED

1. For $x \in R$, $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x =$ [JEE 2000]

(A) e (B) e^{-1} (C) e^{-5} (D) e^5

2. $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ equals [JEE 2001]

(A) $-\pi$ (B) π (C) $\frac{\pi}{2}$ (D) 1

3. Evaluate $\lim_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}, a>0.$ [REE 2001]

4. The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number is

(A) 1 (B) 2 (C) 3 (D) 4

[JEE 2002]

5. If $\lim_{x\to 0}\frac{\sin(nx)[(a-n)nx-\tan x]}{x^2}=0$ $(n>0)$ then the value of 'a' is equal to

(A) $\frac{1}{n}$ (B) n^2+1 (C) $\frac{n^2+1}{n}$ (D) None

[JEE 2003]

6. Find the value of $\lim_{n\to\infty}\left[\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n\right]$. [JEE ' 2004]

7. Let $L=\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}$, $a>0$, If L is finite, then

(A) a = 2 (B) a = 1 (C) $L=\frac{1}{64}$ (D) $L=\frac{1}{32}$

[JEE ' 2009]

8. If $\lim_{x\to 0}\left(1+x\ln\left(1+b^2\right)\right)^{\frac{1}{x}}=2b\sin^2\theta, b>0$ & $\theta\in(-\pi,\pi]$, then the value of is

(A) $\pm\frac{\pi}{4}$ (B) $\pm\frac{\pi}{3}$ (C) $\pm\frac{\pi}{6}$ (D) $\pm\frac{\pi}{2}$

[JEE ' 2011]

9. If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$, then

(A) a = 1, b = 4 (B) a = 1, b = -4 (C) a = 2, b = -3 (D) a = 2, b = 3

[JEE ' 2012]

10. Let α (a) and β (a) be the roots of the equation $\left((1+a)^{1/3}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left((1+a)^{1/6}-1\right)=0, a>-1$, then $\lim\limits_{a\to0^+}\alpha(a)$ & $\lim\limits_{a\to0^+}\beta(a)$ are

(A) -1 & $-\frac{1}{2}$ (B) -1 & $-\frac{3}{2}$ (C) 1 & $-\frac{1}{2}$ (D) -1 & $\frac{3}{2}$

[JEE ' 2012]

11. For a $\in$ R (the set of all real numbers), $a\neq-1$, $\lim\limits_{n\to\infty}\frac{1^a+2^a+\ldots+n^a}{(n+1)^{a-1}\left((na+1)+(na+2)+\ldots+(na+n)\right)}=\frac{1}{60}$, then a =

(A) 5 (B) 7 (C) $-\frac{15}{2}$ (D) $-\frac{17}{2}$

[JEE ' 2013]

ANSWER KEY

EXERCISE - 1(A)

1.	D	2.	D	3.	C	4.	D	5.	D	6.	A	7.	C
8.	B	9.	D	10.	B	11.	A	12.	C	13.	B	14.	A
15.	A	16.	B	17.	D	18.	A	19.	C	20.	D	21.	B
22.	D	23.	C	24.	A	25.	C	26.	C	27.	C	28.	D
29.	D	30.	D	31.	C	32.	A	33.	A	34.	C	35.	A
36.	D	37.	A	38.	D	39.	A	40.	D	41.	D	42.	C
43.	D	44.	A	45.	C	46.	B	47.	C	48.	D	49.	B
50.	A	51.	B	52.	A	53.	B	54.	A	55.	D	56.	B
57.	C	58.	A	59.	D	60.	A	61.	D	62.	A	63.	B
64.	C	65.	A										

EXERCISE - 1(B)

1.	C	2.	D	3.	D	4.	D	5.	B	6.	B	7.	C
8.	A	9.	C	10.	C	11.	D	12.	D	13.	B	14.	D
15.	C	16.	A	17.	B	18.	C	19.	C	20.	A	21.	B
22.	B	23.	A	24.	D	25.	C	26.	C	27.	D	28.	C
29.	A	30.	D	31.	B	32.	D	33.	A	34.	B	35.	D

EXERCISE - 1(C)

1.	D	2.	B	3.	A	4.	D	5.	B	6.	C	7.	C
8.	D	9.	B	10.	B	11.	D	12.	B	13.	D	14.	A
15.	B	16.	D	17.	A	18.	B	19.	C	20	B	21.	C
22.	A	23.	D	24.	D	25.	C	26.	C	27.	A	28.	B
29.	A	30.	B	31.	D	32.	A	33.	D	34.	B	35.	B
36.	C	37.	A	38.	B	39.	D	40.	A				

WINDOW TO JEE MAIN

1.	D	2.	A	3.	B	4.	D	5.	C	6.	B	7.	C
8.	B	9.	D	10.	B	11.	B	12.	A	13.	B	14.	D
15.	D												

EXERCISE - 2 (A)

1	A,D	2	A,B,C,D	3	B,C	4	A,C	5	A, B, D
6	A,B,D	7	A,B,C,D	8	A,B	9	B,C	10	A,B
11	A,B,C	12	C,D	13	A,B,D	14	A,C	15	B,C,D
16	C	17	B	18	B	19	A	20	B
21	A	22	B	23	C	24	D		

25 (A) R; (B) S; (C) P; (D) Q

26 (A) Q; (B) R; (C) P; (D) P

27 (A) S; (B) P ; (C) Q; (D) R

28 (A) S; (B) R; (C) P; (D) Q ; (E) P

29 (A) S; (B) R; (C) Q; (D) P

EXERCISE - 2 (B)

1	2	2	10	3	35	**4**	4	**5**	3
6	8	**7**	2	**8**	1	**9**	4	**10**	9
11	2	12	8	13	3	14	4	15	1

EXERCISE - 3

1	$-3, -3, -3$	**2**	$(\ln a)^n$	**3**	$-1/2$	**4**	$\frac{\sqrt{3}}{2}$
5	$\frac{2L}{3}$	**6**	$e^{-2\pi^2 a^2}$	**7**	$e^{-1/2}$	**8**	$(a_1.a_2.a_3....a_n)$
9	$\frac{\pi}{2}, \frac{\pi}{2\sqrt{2}}$	**10**	$\frac{\pi^2 a^2 + 4}{16a^4}$	**12**	$-\frac{1}{\pi^2}$	**13**	$a - b$
14	$\frac{\pi}{3}$	**15**	1/2				

16 $T(x) = \frac{1}{2}\tan^2\frac{x}{2}.\sin x$ or $\tan\frac{x}{2} - \frac{\sin x}{2}$, $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit $= \frac{3}{2}$

17	$g(x) = \sin x$ and $l = e$	**18**	$\frac{\theta}{\tan\theta}$	**19**	19	**20**	e^{π^2}

22	307	**23**	11	**24**	**(a)** $a_1^{p_1} \cdot a_2^{p_2} \ldots\ldots a_n^{p_n}$; **(b)** a_1 ; **(c)** a_n				
25	$\cot^{-1}\frac{1}{2}$								

WINDOW TO JEE ADVANCED

1	C	**2**	B	**3**	$ln\, a$	**4**	C	**5**	C
6	$1-\frac{2}{\pi}$	**7**	A, C	8	D	9	B	10	A
11	B,D								