Matrices & Determinants

Exercise – 1(C)

Q.1

$A=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}$ gives characteristic equation of A as

$\begin{vmatrix}1-x & 2\\ -1 & 3-x\end{vmatrix}=0$ or $x^2-4x+5=0$

Hence $A^2=4A-5I.$

$\Rightarrow A^3=4A^2-5A=11A-20I$

$\Rightarrow A^4=11A^2-20A=24A-55I$

$\Rightarrow A^5=24A^2-55A=41A-120I$

$\Rightarrow A^6=41A-120I=44A-205I$

Now

$A^6-4A^5+8A^4-12A^3+14A^2$

$=44A-205I-4(41A-120I)+8(24A-55I)-12(11A-20I)+14(4A-5I)$

$=-4A+5I.$

$\Rightarrow \alpha+\beta=1$

Q.2

$|A^3|=125$

$\Rightarrow |A|^3=5^3$

$\Rightarrow |A|=5$

$\Rightarrow \alpha^2-4=5 \Rightarrow \alpha^2=9$

Q.3

$$\begin{bmatrix}1+x & x & x^2\\ x & 1+x & x^2\\ x^2 & x & 1+x\end{bmatrix}=\begin{bmatrix}1+x & x & x^2\\ x & 1+x & x^2\\ x^2 & x & 1+x\end{bmatrix}+\begin{bmatrix}1 & x & x^2\\ 0 & 1+x & x^2\\ 0 & x & 1+x\end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\Delta_1 \qquad + \qquad \Delta_2$$

$$\Delta_1 = x\begin{bmatrix} 1+x & x & x^2 \\ 1 & 1+x & x^2 \\ x^2 & x & 1+x \end{bmatrix}$$

$\Delta_2 = (1+x)^2 - x^3$

$= 1 + x^2 + 2x - x^3$

$\therefore \Delta = \Delta_1 + \Delta_2$

$= x + x^2 + x^5 + 1 + x^2 + 2x - x^3$

$\Delta = x^5 - x^3 + 2x^2 + 3x + 1$

$\therefore$ Comparing. $\underline{\underline{t = 3}}$

Q.4

$\frac{1}{a} = A + (p-1)d$

$\frac{1}{b} = A + (q-1)d$

$\frac{1}{c} = A + (r-1)d$

$\frac{1}{a} - \frac{1}{b} = (p-q)d \Rightarrow (b-a) = (p-q)\ abd$

$\frac{1}{b} - \frac{1}{c} = (q-r)d \Rightarrow c - 3 = (q-r)\ bcd$

$$\therefore \begin{bmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c(b-a) & a(c-b) & ab \\ p-q & q-r & r \\ 0 & 0 & 1 \end{bmatrix} c_1 - c_2\ ;\ c_2 - c_3$$

$$= \begin{bmatrix} c(p-q)abd & a(q-r)bcd & ab \\ (p-q) & (q-r) & r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (p-q)(q-r)\,ab \begin{bmatrix} cd & cd & b \\ 1 & 1 & r \\ 0 & 0 & 1 \end{bmatrix}$$

$= (p - q)(q - r)\, ab\,(cd - cd) = 0$

Q.5

$AB = A \Rightarrow BAB = BA \Rightarrow B^2 = BA \Rightarrow B^2 = B$

$BA = B \Rightarrow ABA = AB \Rightarrow A^2 = AB \Rightarrow A^2 = A$

Hence $k = l = 1$. Further

$$(A+B)^2 = A^2 + AB + BA + B^2 = 2(A+B)$$
$$\Rightarrow (A+B)^3 = 2(A+B)^2 = 4(A+B)$$

Hence $m = 4$.

$k + l + m = 6$.

Q.6

$$f(x) = \begin{bmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{bmatrix}$$

$$f'(x) = 4\underbrace{\begin{bmatrix} (x-a)^3 & (x-a)^3 & 1 \\ (x-b)^3 & (x-b)^3 & 1 \\ (x-c)^3 & (x-c)^3 & 1 \end{bmatrix}}_{0} + 3\begin{bmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{bmatrix} + \underbrace{\begin{bmatrix} (x-a)^4 & (x-a)^3 & 0 \\ (x-b)^4 & (x-b)^3 & 0 \\ (x-c)^4 & (x-c)^3 & 0 \end{bmatrix}}_{0}$$

$$\therefore f'(x) = 3\begin{bmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{bmatrix}$$

$\therefore\ \lambda = 3$

Q.7

(i) $\ln a = \lim_{x\to 0^+} x^x - \lim_{x\to 0^+} 1$

$= e^{\lim_{x\to 0^+} x\ln x} - 1 = 0$

$\Rightarrow \boxed{a = 1}$

(ii) $b = \lim_{x\to 0^+} (\cot x)^{\sin x}$

$\ln b = \lim_{x\to 0^+} \left(\sin x \ln(\cot x)\right) =$

$= \lim_{x\to 0^+} \frac{\ln(\cot x)}{\operatorname{cosec} x}$

$= \lim_{x\to 0^+} \frac{1}{(\cot x)} \cdot \frac{(-\operatorname{cosec}^2 x)}{-\frac{1}{\sin^2 x}\cdot(\cos x)}$ (L' Hospital Rule)

$= \underset{x\to 0}{\text{lt}} \frac{\sin x}{\cos^2 x} = 0$

$\Rightarrow \boxed{b = 1}$

(iii)

$c = \lim_{x\to 0^+} (\sec x)^{\operatorname{cosec} x} \Rightarrow \ln c = \lim_{x\to 0^+} \left(\operatorname{cosec} x \ln(\sec x)\right)$

$= \lim_{x\to 0^+} \frac{\ln(\sec x)}{\sin x}$

$= \lim_{x\to 0^+} \frac{1}{\sec x}\cdot\frac{\sec x \tan x}{\cos x}$ (L' Hospital Rule)

$= 0$

$\Rightarrow \boxed{c = 1}$

(iv) $d = \lim_{x\to 0^+} \frac{\ln(\sec x)}{x^2}$

$= \lim_{x\to 0^+} \frac{1}{\sec x}\cdot\frac{\sec x \tan x}{2x}$ (L' Hospital Rule)

$\Rightarrow d = \frac{1}{2}$

$\therefore A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore \operatorname{adj}(A) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\therefore |\text{Adj (Adj(A))}| = \text{adj (adj(A))} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2$

Q.8

$AB = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore (AB)^r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \sum_{r=1}^{50} Tr\left((AB)^r C_r\right)$$

$$= \sum_{r=1}^{50} Tr\left(C_r\right)$$

$$= \sum_{r=1}^{50} r \cdot 3^r + (r-1)3^r = 2\sum_{r=1}^{50} r \cdot 3^r - \sum_{r=1}^{50} 3^r$$

Q.9

$$AB = (AB)^{-1}$$

$$(AB)^2 = I$$

$\Rightarrow$ AB is involutory.

$$\therefore AB = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$$

$$= \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x - 2 & 5x \end{bmatrix}$$

$$AB^2 = \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x - 2 + 5x(10x-2) & 25x^2 \end{bmatrix}$$

$= I$

$$\therefore 5x = 1 \text{ and } 25x^2 = 1$$

$$\Rightarrow \boxed{x = \frac{1}{5}}$$

Now,

$$\text{Tr}\ (AB + (AB)^2 + (AB)^2 + ... + (AB)^{100})$$

$$= \text{Tr}\left[(AB) + I + (AB + I) + ...(AB + I)\right]$$

$= 50\ (10x + 4)$

$$= 50\left(10 \times \frac{1}{5} + 4\right)$$

$= 300$

$$\therefore \frac{\text{Tr}()}{100} = 3$$

Q.10

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \Rightarrow |\,A\,| = 1$$

Now $B = A + 4A^2 + 6A^3 + 4A^4 + A^5 = A(A + I)^4$.

$|\,B\,| = |\,A\,|\,|\,A + I\,|^4 = 1.$

$$A + I = \begin{bmatrix} 1 + \cos\alpha & \sin\alpha \\ -\sin\alpha & 1 + \cos\alpha \end{bmatrix} \Rightarrow |\,A + I\,| = (1 + \cos\alpha)^2 + \sin^2\alpha = 2 + 2\cos\alpha$$

Hence $2 + 2\cos\alpha = 1$ or $\cos\alpha = -\dfrac{1}{2}$.

Therefor 4 values of α are possible.

Q.11

$$M^2 = 0 \Rightarrow \begin{bmatrix} a & -360 \\ b & c \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & -360 \\ b & c \end{bmatrix}\begin{bmatrix} a & -360 \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2 - 360b & -360a - 360c \\ ab + bc & -360 + c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow a^2 - 360b = 0 \qquad .....(1)$$

$\Rightarrow -360(a+c)=0$ .....(2)

$\Rightarrow b(a+c)=0$ .....(3)

$\Rightarrow -360b+c^2=0$ ......(4)

$\Rightarrow a^2=360b$

$\Rightarrow a+c=0 \Rightarrow a=-c \Rightarrow a^2=c^2$

$\Rightarrow c^2=360b$

Q.12

$$M^2-\lambda M-I_2=0 \Rightarrow \begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}5 & 8\\ 8 & 13\end{bmatrix}-\begin{bmatrix}\lambda & 2\lambda\\ 2\lambda & 3\lambda\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$\Rightarrow 5-\lambda-1=0$

$\Rightarrow \lambda=4$

Q.13

$$A=\begin{bmatrix}2 & 4 & 5\\ 4 & 8 & 10\\ -6 & -12 & -15\end{bmatrix}$$

$$A=\begin{bmatrix}2 & 4 & 5\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} R_2 \to 2R_1\ ;\ R_3 \to -3(R_1)$$

$\therefore$ Only one independent row

$\therefore$ Rank = 1

Q.14

$A^3=A\cdot A^2=A(5A-7I)$

$=5A^2-7A$

$=5(5A-7I)-7A$

$A^4 = A\cdot A^3 = A(18A-35I)$

$=18A^2-35A$

$= 18(5A-7I)-35A$

$= 55A-126I$

Finally $A^5 = A\cdot A^4$

$= A(55A-126I)$

$=55A^2-126A$

$= 149A-385I$

$\therefore$ a = 149 & b = – 385

$$\therefore \frac{4a-6b}{1453} = \frac{4(149)+6(385)}{1453} = \frac{2906}{1453} = 2$$

Q.15

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $X^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

$$X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$$

x’ A x = $= \left[a_{11}x_1^{\ 2} + a_{22}x_2^{\ 2} + a_{33}x_3^{\ 2} + \left(a_{12}+a_{21}\right)x_1x_2 + \left(a_{23}+a_{32}\right)x_2x_3 + \left(a_{13}+a_{31}\right)x_1x_{13}\right]$

This is true for every x_1 , x_2 , x_3, then

$a_{11} = a_{22} = a_{33} = 0$ and $a_{12}+a_{21} = a_{23}+a_{32} = a_{13}+a_{31} = 0$.

$\because a_{12} = 145 \quad \Rightarrow \quad a_{21} = -145$

$a_{23} = -2008 \Rightarrow a_{32} = 2008$

$a_{31} = -182 \quad \Rightarrow a_{31} = 182$

$\therefore \frac{a_{32}}{1004} = \frac{2008}{1004} = 2$

Q.16

$$A(\text{adj}A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A(\text{adj}A) = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|I$$

$\therefore |A| = 10$

$$\therefore \frac{|A|^3}{200} = \frac{1000}{200} = 5$$

Q.17

AB = BA

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$

$b + 2d = 2a + 4b$

$2(d-a) = 3b$

$\boxed{a-d = \frac{-3b}{2}}$

$3a + 4c = c + 3d$

$3(a-d) = -3c$

$\boxed{a-d=-c}$ ……….(2)

$$\therefore \left|\frac{a-d}{3b-c}\right| = \frac{|-c|}{|c|} = 1$$

Q.18

$AA^T = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow a + 4 + 2b = 0$ ....(1) $\Rightarrow a + 2b + 4 = 0$

$\Rightarrow 2a + 2 - 2b = 0$ ....(2) $\Rightarrow a - b + 1 = 0$

$(1) - (2)$

$3b + 3 = 0$

$\boxed{b = -1}$

$a^2 + 4 + b^2 = 9$ ....(4)

$\therefore a^2 + 4 + b^2 = 9 \Rightarrow a^2 + 4 + 1 = 9 \Rightarrow a^2 = 4 \Rightarrow a^2 = \pm 2$

$\boxed{a = -2}$ from (1)

Q.19

$$\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} = 10A^{-1}$$

$$\Rightarrow \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

$\Rightarrow -5 + \alpha = 0$

$\Rightarrow \alpha = 5$ ....(equating element a_{21})

Q.20

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}_{3\times 3} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + x_2 + x_3 & y_1 + y_2 + y_3 \\ 2x_1 + 5x_2 + 7x_3 & 2y_1 + 5y_2 + 7y_3 \\ 2x_1 + x_2 - x_3 & 2y_1 + y_2 - y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 + x_3 = 9$$

$$\Rightarrow y_1 + y_2 + y_3 = 2$$

$$\therefore x_1 + x_2 + x_3 - y_1 - y_2 - y_3 = 9 - 2 = 7$$