Binomial Theorem

Exercise – 2

Q.1 [A]

$(1+x)^n = C_0 + C_1x + C_2x^2 + ...C_nx^n$

Differentiating w.r.t. x,

$\Rightarrow n(1+x)^{n+1} = 1C_1 + 2C_2x + 3C_3x^2 + ... + (n)C_nx^{n-1}$

Multiplying with x on both sides,

$\Rightarrow nx(1+x)^{n-1} = C_1x + 2C_2x^n + 3C_3x^3 + ... + nC_nx^n$

Again differentiating w.r.t. x,

$\Rightarrow n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} = 1^2C_1 + 2^2C_2x + 3^2C_3x^2 + ... + n^2C_nx^{n-1}$

Now, substituting x = 1 in above equation,

$\Rightarrow n2^{n-1} + n(n-1)2^{n-2} = 1^2C_1 + 2^2C_2 + 3^2C_3 + ... + n^2C_n$

$\Rightarrow n(n+1)2^{n-2}$

Q.2 [D]

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$

Multiplying with x,

$\Rightarrow x(1+x)^n = {}^nC_0x^1 + {}^nC_1x^2 + {}^nC_2x^3 + ... + {}^nC_nx^{n+1}$

Now, integrating w.r.t. x, from x = 0 to x = 1,

$$\Rightarrow \int_0^1 \left({}^nC_0x^1 + {}^nC_1x^2 + {}^nC_2x^3 + ... + {}^nC_nx^{n+1}\right)dx = \int_0^1 x(1+x)^n\,dx$$

$$\Rightarrow \frac{{}^nC_0}{2} + \frac{{}^nC_1}{3} + \frac{{}^nC_2}{4} + ... + \frac{{}^nC_n}{n+2} = I$$

$$\Rightarrow I=\int_0^1 (1+x)^n\, x dx \qquad 1 + x = t \text{ ; } dx = dt$$

$$\Rightarrow \int_1^0 t^n (t-1) dt$$

$$\Rightarrow \int_1^0 \left(t^{n+1}-t^n\right) dt$$

$$\Rightarrow \left[\frac{t^{n+2}}{n+2}-\frac{t^{n+1}}{n+1}\right]_1^0$$

$$\Rightarrow \frac{1}{n+1}-\frac{1}{n+2}=\frac{1}{(n+1)(n+1)}$$

So,

$$\Rightarrow \frac{{}^nC_0}{2}+\frac{{}^nC_1}{3}+\frac{{}^nC_2}{4}+..........+\frac{{}^nC_n}{n+2}=\frac{1}{(n+1)(n+2)}$$

Hence,

$$\Rightarrow \frac{{}^nC_1}{3}+\frac{{}^nC_2}{4}+\frac{{}^nC_3}{5}+............+\frac{{}^nC_n}{n+2}=\frac{1}{(n+1)(n+2)}-\frac{1}{2}$$

Q.3 [A]

$$(1-x)^n = C_o - C_1x + C_2x^2 - C_3x^3 + C_4x^4 + ... + (-1)^n C_n x^n$$

Differentiating w.r.t. x,

$$\Rightarrow n(1-x)^{n-1} = -\left[C_1 - 2C_2x + 3C_3x^2 - 4C_4x^3 + 5C_5x^4 + ... + (-1)^{n-1} n C_n x^{n-1}\right]$$

substituting x = 1,

$$\Rightarrow C_1 - 2C_2 + 3C_3 - 4C_4 + 5C_5 .... (-1)^{n-1} n C_n x^{n-1} = 0$$

Q.4

If n is even natural number,

$$\Rightarrow S = 3C_0 - 5C_1 + 7C_2 + ... + (2n+3)C_n \quad ....(i)$$

$$\Rightarrow S = (2n+3)C_n - (2n+1)C_{n-1} + (2n-1)C_{n-2} + ... + 3C_0 \quad ....(ii)$$

$$\Rightarrow S = (2n+3)C_0 - (2n+1)C_1 + (2n-1)C_2 + ... + 3C_n \quad ....(iii)$$

Adding (i) & (iii),

$$\Rightarrow 2S = (2n+6)[C_0 - C_1 + C_2 - C_3 + C_4 - C_5 .... + C_n]$$

$$\therefore S = 0 \qquad (\because C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ...)$$

If n is odd natural number,

$$\Rightarrow S = 3C_0 - 5C_1 + 7C_2 - .... - (2n+3)C_n \quad ....(iv)$$

$$\Rightarrow S = -(2n+3)C_0 + (2n+1)C_1 - (2n-1)C_2 .... + 3C_0 \quad ....(v)$$

Adding (iv) & (v)

$$\Rightarrow 2S = -2n(C_0 - C_1 + C_2 - C_3 ....)$$

$$\Rightarrow S = 0$$

Q.6 [B]

$$\sum_{r=0}^{n} r^3 \left(\frac{n+1-r}{r}\right)^2 = \sum_{r=0}^{n} r(n+1)^2 + \sum_{r=0}^{n} r^3 - \sum_{r=0}^{n} 2(n+1)r^2$$

$$\Rightarrow (n+1)^2 \left(\sum_{r=0}^{n} r\right) + \left(\sum_{r=0}^{n} r^3\right) - 2(n+1)\left(\sum_{r=0}^{n} r^2\right)$$

$$\Rightarrow (n+1)^2 \frac{n(n+1)}{2} + \left[\frac{n(n+1)}{2}\right]^2 - \frac{2(n+1)(n+1)n(2n+1)}{6}$$

$$\Rightarrow \frac{n(n+1)^2(n+2)}{12}$$

Q.17 [A]

The coefficient of x^n in

$$\Rightarrow \left(1+x+2x^2+3x^3+4x^4+............+nx^n\right)\left(nx^n+(n-1)x^{n-1}+.......+2x^2+x+1\right)$$

is $(1\cdot n)+1\cdot(n-1)+2(n-2)+3(n-3)+.........+(n-2)\cdot 2+(n-1)\cdot 1+n-1$

$$\Rightarrow 2n+\sum_{r=1}^{n-1} r(n-r)$$

$$\Rightarrow 2n+\frac{n(n-1)n}{2}-\frac{(n-1)n(2n-1)}{6}$$

$$\Rightarrow 2n+\frac{n(n-1)}{6}[3n-2n+1]$$

$$\Rightarrow 2n+\frac{n(n^2-1)}{6}$$

$$\Rightarrow \frac{n(n^2+11)}{6}$$

Q.20 [B]

$$3^{37}=\left(3^4\right)^9\times 3$$

$$\Rightarrow (80+1)^9\times 3$$

$$\Rightarrow 3(1+80)^9$$

$$\Rightarrow 3\left(1+{}^9C_1 80+{}^9C_2 80^2+{}^9C_3 80^3+{}^9C_4 80^4+............+80^9\right)$$

$$\Rightarrow 3 + 80\ k,\ k\in N$$

Hence, the remainder when 3^{37} is divided by 80 is 3.

Q.21 [C]

$${}^{50}C_{25}=\frac{50!}{25!\ 25!}$$

$$\Rightarrow \frac{5\times 49\times 48\times 47\times ....\times 27\times 26\times 25\times 24\times ....\times 1}{(1\times 2\times 3\times 4\times ....\times 24\times 25)}$$

Exponent of 3 in this number

$\Rightarrow$ (16 + 5 +1) – [(8 + 2) + (8 + 2)]

$\Rightarrow 2$

Exponent of 2 in this number

$\Rightarrow$ (25 + 12 + 6 +3 + 1) – 2 (12 + 6 + 3 +1)

$\Rightarrow 3$

Hence, highest power of 18, i.e. $2^1 \times 3^2$ in ${}^{50}C_{25}$ is 1.

Q.22 [C]

$(101)^{100} - 1 = (1+100)^{100-1}$

$\Rightarrow \left[1 + {}^{100}C_1(100) + {}^{100}C_2(100)^2 + {}^{100}C_3(100)^3 + ... + 100^{100}\right] - 1$

$\Rightarrow 100^2\left[1 + {}^{100}C_2 + {}^{100}C_3(100) + ... + (100)^{98}\right]$

Q.24 [C]

$99^n + 1$

$\Rightarrow (100-1)^n + 1$

$\Rightarrow 100^n - {}^nC_1 100^{n-1} + {}^nC_2(100)^{n-2} - {}^nC_3(100)^{n-3} .... + {}^nC_{n-1}(100)^1(-1)^{n-1} + (-1)^n + 1$

$\Rightarrow 10^2\left(100^{n-2} - {}^nC_1(100)^{n-3} + {}^nC_2(100)^{n-4} - {}^nC_3(100)^{n-5} .... + {}^nC_{n-1}\right)$

$\Rightarrow 100\text{ k}$, where $k \in N$

Q.25 [C]

$3^{2003} = \left(3^3\right)^{667} \cdot 3^2$

$\Rightarrow 9(28-1)^{667}$

$\Rightarrow 9\left[28^{667} - {}^{667}C_1 28^{666} + ... + 28 \times {}^{667}C_{666} - 1\right]$

$\Rightarrow 28k - 9$, $k \in N$

$\Rightarrow 28\ (k - 1) + (28 - 9)$

$\Rightarrow 28\ (k - 1) + 19$

Hence, $\left\{\frac{3^{2003}}{28}\right\} = \frac{19}{28}$

26. [A]

$$P = \left(8 + 3\sqrt{7}\right)^n = {}^nC_0 8^n + {}^nC_1 8^{n-1}\left(3\sqrt{7}\right)^1 + {}^nC_2 8^{n-2}\left(3\sqrt{7}\right)^2 + \ldots\ldots\ldots\ldots$$

$$\Rightarrow f' = \left(8 - 3\sqrt{7}\right)^n = {}^nC_0 8^n - {}^nC_1 8^{n-1}\left(3\sqrt{7}\right)^1 + {}^nC_2 8^{n-2}\left(3\sqrt{7}\right)^2 + \ldots\ldots\ldots\ldots$$

Now, since $8 - 3\sqrt{7} = \frac{1}{8 + 3\sqrt{7}}$

$\Rightarrow 8 - 3\sqrt{7} \in (0,1)$

$\Rightarrow \left(8 - 3\sqrt{7}\right)^n \in (0,1)$

$\Rightarrow 0 < f' < 1$

$$\Rightarrow P + f' = 2\left[{}^nC_0 8^n + {}^nC_2 8^{n-2}\left(3\sqrt{7}\right)^2 + {}^nC_4 8^{n-4}\left(3\sqrt{7}\right)^4 + \ldots\ldots\ldots\ldots\right]$$

$\Rightarrow 2k,\ k \in N$

$\Rightarrow [P] + f + f' = 2k,\ k \in N$

$\Rightarrow 0 < f < 1 \ \&\ 0 < f' < 1$

$\Rightarrow 0 < f + f' < 2$

Since 2k & [P] are integers, $(f + f')$ has to be an integer & $0 < f + f' < 2$

$\Rightarrow f + f' = 1$

$$\Rightarrow \therefore P(1 - f) = P \cdot f' = \left(8 + 3\sqrt{7}\right)^n \left(8 - 3\sqrt{7}\right)^n = 1^n = 1$$

Q.28 [C]

$\left(1+10^{-4}\right)^{10^4}$

$\Rightarrow 1+\left(10^{-4}\right)10^4 + {}^{\left(10^4\right)}C_2\left(10^{-4}\right)^2 + {}^{\left(10^4\right)}C_3\left(10^{-4}\right)^3 + ...$

$\Rightarrow 1 + 1 + f$

where $0 < f < 1$.

Q.31

coefficient of x^n in $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+........+\frac{x^n}{n!}\right)\left(\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}+.........+\frac{x^2}{2!}+\frac{x}{1!}+1\right)$

is $\frac{1}{1!\,n!}+\frac{1}{1!(n-1)!}+\frac{1}{2!(n-2)!}+........+\frac{1}{(n-1)!\,1!}+\frac{1}{n!1!}$

$\Rightarrow \sum_{r=0}^{n}\frac{1}{r!(n-r)!}$ $\qquad (\because 1! = 0!)$

$\Rightarrow \frac{1}{n!}\sum_{r=0}^{n}\left(\frac{n!}{r!(n-r)!}\right)=\frac{2^n}{n!}$

Q.33 [D]

${}^{21}C_6 + {}^{21}C_5 + {}^{22}C_5 + {}^{23}C_5 + {}^{24}C_5 + {}^{25}C_5 + {}^{26}C_5$

$\Rightarrow {}^{22}C_6 + {}^{22}C_5 + {}^{23}C_5 + ...$

$\Rightarrow {}^{23}C_6 + {}^{23}C_5 + {}^{24}C_5 + {}^{25}C_5 + ...$

$\Rightarrow {}^{24}C_6 + {}^{24}C_5 + ...$

$\Rightarrow {}^{26}C_6 + {}^{26}C_5 = {}^{27}C_6 = {}^{n}C_6$

$\Rightarrow n = 27$

Q.34 [A]

$(101)^{100} - 1$

$\Rightarrow (1+100)^{100} - 1$

$\Rightarrow \left(1 + {}^{100}C_1 (100)^1 + {}^{100}C_2 (100)^2 + {}^{100}C_3 (100)^3 + ... + (100)^{100}\right) - 1$

$\Rightarrow (100)^2 \left[1 + {}^{100}C_2 + {}^{100}C_3 (100) + ... + (100)^{98}\right]$

$\Rightarrow 10000k,\ k \in N$

Q.35 [C]

Coefficient of x^{10} in $(1 - 2x + 3x^2 - 4x^3 + ...\infty)(1 - 2x + 3x^2 - 4x + ...\infty)$

$\Rightarrow$ 11 (1) + (– 10) (– 2) + (+9) (3) …………..+ (1) (11)

$\Rightarrow \sum_{r=1}^{11} r(12 - r)$

$\Rightarrow 12\left(\sum_{r=1}^{11} r\right) - \left(\sum_{r=1}^{11} r^2\right)$

$\Rightarrow 12 \times \frac{11 \times 2}{2} - \frac{11 \times 12 \times 23}{6}$

$\Rightarrow \frac{11 \times 12 \times 13}{1 \times 2 \times 3} = \frac{(13)!}{10!\,3!} = {}^{13}C_{10}$

Q.36 [A]

$a^6 + b^6 + 3a^4b^{12} + 3a^2b^4 = a^6 + b^6 + 2a^3b^3$

$\Rightarrow 3a^4b^2 - 2a^3b^3 + 3a^2b^4 = 0$

Dividing with $a^3\, b^3$,

$\Rightarrow 3\left(\frac{a}{b}\right) - 2 + 3\left(\frac{b}{a}\right) = 0$

$\Rightarrow \left(\frac{a}{b} + \frac{b}{a}\right) = \frac{2}{3}$

Q.38 [B]

Coefficient of $x^m = A = {}^{m+n}C_m$

Coefficient of $x^n = B = {}^{m+n}C_n = {}^{m+n}C_m$

Hence, A = B

Q.39 [A]

$(1+i)^{100} = 1 + {}^{100}C_1 i + {}^{100}C_2 (i)^2 + {}^{100}C_3 (i)^3 + {}^{100}C_4 (i)^4 + ...$

$\Rightarrow \left(1 - {}^{100}C_2 + {}^{100}C_4 - {}^{100}C_6 + ... + {}^{100}C_{100}\right) + \left({}^{100}C_1 i - {}^{100}C_3 i + {}^{100}C_5 i - {}^{100}C_7 i + ... - {}^{100}C_{99} i\right)$

$\Rightarrow \therefore$ Number of purely imaginary terms = 50.