Electric Potential due to a Point Charge, a Dipole and a System of Charges:

(a) Electric potential due to a point charge:

The electric potential at a point A is the amount of work done per unit positive charge, which is displaced from ∞ to point A.

![Diagram of electric potential due to a point charge](image)

Let M be an intermediate point on this path where OM = x. The electrostatic force on a unit positive charge at M is of magnitude

\[ F = \frac{1}{4\pi\varepsilon_0} \frac{q}{x^2} \]  

\[ \text{...............(i)} \]

It is directed away from O, along OM.
For infinitesimal displacement dx from M to N, the amount of work done is given by dW=Fdx

\[ \text{..................(ii)} \]

The negative sign appears as the displacement is directed opposite to that of the force.

\[ \therefore \text{Total work done in displacing the unit positive charge from } \infty \text{ to point A is given by;} \]

\[ W = \int_{\infty}^{r} -Fdx = \int_{\infty}^{r} -\frac{1}{4\pi\varepsilon_0} \frac{q}{x^2} \, dx \]

\[ = -\frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{x} \right]_{\infty}^{r} \]

\[ = -\frac{q}{4\pi\varepsilon_0} \left[ \frac{-1}{r} \right] \quad \left( \because \int x^{-2} \, dx = \frac{-1}{x} \right) \]

\[ = \frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{r} - \frac{1}{\infty} \right] \quad \left( \because \frac{1}{\infty} = 0 \right) \]

\[ W = \frac{q}{4\pi\varepsilon_0 r} \quad \text{..................(iii)} \]

By definition this is the electrostatic potential at A due to charge q.

\[ \therefore V = W = \frac{q}{4\pi\varepsilon_0 r} \quad \text{..................(iv)} \]
A positively charged particle produces a positive electric potential and a negatively charged particle produces a negative electric potential.

\[ V = \frac{q}{\infty} = 0 \]

This shows that the electrostatics potential is zero at infinity.

Equation (iv) shows that for any point at a distance \( r \) from the point charge \( q \), the value of \( V \) is the same and is independent of the direction of \( r \).

Hence electrostatic potential due to a single charge is spherically symmetric.

**Variation of electric potential** \( V \propto \frac{1}{r} \) and **electric field** \( E \propto \frac{1}{r^2} \) vary with \( r \), the distance from the charge.

(b) **Electric potential due to an electric dipole**:

Consider an electric dipole AB consisting of two charges +q and -q separated by a finite distance 2l. Its dipole moment is \( p \), of magnitude \( p = q \times 2l \), directed from -q to +q.

Let C be any point near the electric dipole at a distance \( r \) from the centre O inclined at an angle \( \theta \) with axis of the dipole.

\( r_1 \) and \( r_2 \) the distances of point C from charges +q and -q, respectively.

Potential at C due to charge +q at A is,

\[ V_1 = \frac{+q}{4\pi \varepsilon_0 r_1} \]
Potential at C due to charge \(-q\) at B is,

\[
V_2 = \frac{-q}{4\pi\varepsilon_0 r_2}
\]

The electrostatic potential is the work done by the electric field per unit charge, \(V = \frac{V}{q}\)

The potential at C due to the dipole is,

\[
V_C = V_1 + V_2 = \frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{r_1} - \frac{1}{r_2} \right]
\]

By geometry,

\[
\begin{align*}
    r_1^2 &= r^2 + \ell^2 - 2r\ell\cos\theta \\
    r_2^2 &= r^2 + \ell^2 + 2r\ell\cos\theta \\
    r_1^2 &= r^2 \left(1 + \frac{\ell^2}{r^2} - 2\frac{\ell}{r}\cos\theta\right) \\
    r_2^2 &= r^2 \left(1 + \frac{\ell^2}{r^2} + 2\frac{\ell}{r}\cos\theta\right)
\end{align*}
\]

For a short dipole, \(2\ell \ll r\) and

If \(r >> \ell\) \(\frac{\ell}{r}\) is small \(\therefore\) \(\frac{\ell^2}{r^2}\) can be neglected

\[
\begin{align*}
    \therefore r_1^2 &= r^2 \left(1 - 2\frac{\ell}{r}\cos\theta\right) \\
    r_2^2 &= r^2 \left(1 + \frac{2\ell}{r}\cos\theta\right) \\
    \therefore r_1 &= r \left(1 - \frac{2\ell}{r}\cos\theta\right)^{\frac{1}{2}} \\
    r_2 &= r \left(1 + \frac{2\ell}{r}\cos\theta\right)^{\frac{1}{2}} \\
    \therefore \frac{1}{r_1} &= \frac{1}{r} \left(1 - \frac{2\ell}{r}\cos\theta\right)^{\frac{1}{2}} \text{ and}
\end{align*}
\]
\[
\frac{1}{r_3} = \frac{1}{r} \left(1 + \frac{2 \ell}{r} \cos \theta \right)^{\frac{1}{2}}
\]

\[
\therefore V_c = V_1 + V_2 = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{r} \left(1 - \frac{2 \ell}{r} \cos \theta \right)^{\frac{1}{2}} - \frac{1}{r} \left(1 + \frac{2 \ell}{r} \cos \theta \right)^{\frac{1}{2}} \right]
\]

Using binomial expansion, \((1 + x)^n = 1 + nx, x \ll l\) and retaining terms up to the first order of \(\frac{l}{r}\) only, we get

\[
V_c = \frac{q}{4\pi \varepsilon_0} \frac{1}{r} \left[ \left(1 + \frac{\ell}{r} \cos \theta \right) - \left(1 - \frac{\ell}{r} \cos \theta \right) \right]
\]

\[
= \frac{q}{4\pi \varepsilon_0} \frac{1}{r} \left[ 1 + \frac{\ell}{r} \cos \theta - 1 + \frac{\ell}{r} \cos \theta \right]
\]

\[
= \frac{q}{4\pi \varepsilon_0} \frac{2 \ell}{r} \cos \theta
\]

\[
\therefore V_c = \frac{1}{4\pi \varepsilon_0} \frac{p \cos \theta}{r^2} \quad (\because p = q \times 2\ell)
\]

Electric potential at C, can also be expressed as,

\[
V_c = \frac{1}{4\pi \varepsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3}
\]

\[
V_c = \frac{1}{4\pi \varepsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^2}, \quad \left( \vec{r} = \frac{\vec{r}}{r} \right)
\]

Where, \(\hat{r}\) is a unit vector along the position vector, \(\vec{OC} = \hat{r}\)

Case (i) Potential at an axial point, \(\theta = 0^\circ\) (towards +q) or \(180^\circ\) (towards – q)

\[
V_{axial} = \pm \frac{1}{4\pi \varepsilon_0} \frac{p}{r^2}
\]

i.e. This is the maximum value of the potential.

Case (ii) Potential at an equatorial point, \(\theta = 90^\circ\) and \(V = 0\)

Hence, the potential at any point on the equatorial line of a dipole is zero.
This is the minimum value of the magnitude of the potential of a dipole.

**Note:**
The plane perpendicular to the line between the charges at the midpoint is an equipotential plane with potential zero. The work done to move a charge anywhere in this plane (potential difference being zero) will be zero.

**(c) Electrostatics potential due to a system of charges:**

Consider a system of charges \( q_1, q_2, \ldots, q_n \) at distances \( r_1, r_2, \ldots, r_n \) respectively from point \( P \).

The potential \( V_1 \) at \( P \) due to the charge \( q_1 \) is

\[
V_1 = \frac{1}{4\pi \varepsilon_0} \frac{q_1}{r_1}
\]

Similarly the potentials \( V_2, V_3, \ldots, V_n \) at \( P \) due to the individual charges \( q_2, q_3, \ldots, q_n \) are given by

\[
V_2 = \frac{1}{4\pi \varepsilon_0} \frac{q_2}{r_2}, \quad V_3 = \frac{1}{4\pi \varepsilon_0} \frac{q_3}{r_3}, \quad V_n = \frac{1}{4\pi \varepsilon_0} \frac{q_n}{r_n}
\]

By the superposition principle, the potential \( V \) at \( P \) due to the system of charges is the algebraic sum of the potentials due to the individual charges.

\[
\therefore V = V_1 + V_2 + \ldots + V_n = \frac{1}{4\pi \varepsilon_0} \left( \frac{q_1}{r_1} + \frac{q_2}{r_2} + \ldots + \frac{q_n}{r_n} \right)
\]

Or,

\[
V = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i}
\]
Equipotential Surfaces:

An equipotential surface is that surface, at every point of which the electric potential is the same. The potential \( V \) for a single charge \( q \) is given by:

\[
V = \frac{1}{4\pi \varepsilon_0} \frac{q}{r}
\]

If \( r \) is constant then \( V \) will be constant.

(i) **For a single point charge:**

Equipotential surfaces of single point charge are concentric spherical surfaces centered at the charge.

(ii) **For a line charge:**

The shape of equipotential surface is cylindrical.

(iii) **For a uniform electric field :**

(iv) **For a dipole :**

(v) **For two identical positive charge :**
(vi) **Between two plane metallic sheet**

![Diagram of two plane metallic sheets with equipotential surfaces]

(vii) **When one of the sheet is replaced by a charged metallic sphere.**

Like the lines of force, the equipotential surfaces give a visual picture of both the direction and the magnitude of electric field in a region of space.

![Diagram of a charged metallic sphere with equipotential surfaces]

**Note:**

(i) By definition the potential difference between two points P and Q is the work done per unit positive charge displaced from Q to P.

\[ V_p - V_q = W_{Qp} \]

If points P and Q lie on an equipotential surface, \( V_p = V_q \)

\[ W_{Qp} = 0 \]

Thus, no work is required to move a test charge along an equipotential surface.
(ii) If $dx$ is the small distance over the equipotential surface through which unit positive charge is carried then

$$dW = E \cdot dx = E \cdot dx \cdot \cos \theta = 0$$

$\cos \theta = 0$ or $\theta = 90^\circ$

i.e. $E \perp dx$

Hence electric field intensity $E$ is always normal to the equipotential surface i.e., for any charge distribution, the equipotential surface through a point is normal to the electric field at that point.

(iii) If the field is not normal, it would have a nonzero component along the surface. So to move a test charge against this component work would have to be done. But by the definition of equipotential surfaces, there is no potential difference between any two points on an equipotential surface and hence no work is required to displace the charge on the surface. Therefore, we can conclude that the electrostatic field must be normal to the equipotential surface at every point, and vice versa.

(iv) Equipotential surfaces do not intersect each other as it gives two directions of electric fields at intersecting point which is not possible.