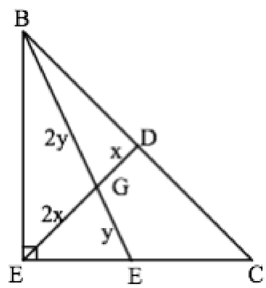
NMTC IX,X-2018
Bhaskara contest -Junior Level

November 5, 2018

1. ABC is a right triangle with BC hypotenuse. The medians drawn to BC and AC are perpendicular to each other. If AB has length 1 cm , find the area of triangle ABC.
   Solution: Let G be the centroid . Let $AG = 2BG = 2x$ and $BG = 2CG = 2y$.
   Let $E = M_{AC}$ and $D = M_{BC}$



   Then in $\triangle AGE$, $AE = \sqrt{4x^2 + y^2}$ and in $\triangle AGB$ we get

$$4(x^2 + y^2) = 1 - - - -(1)$$

   Also in $\triangle BGD$, $BD = \sqrt{4y^2 + x^2} = AD = 3x$

$$\therefore 4y^2 = 8x^2$$
$$y = x\sqrt{2}$$

   From (1) we get

$$x^2 = \frac{1}{6}$$

   $\therefore AE = \sqrt{4x^2 + y^2} = x\sqrt{6} = 1$
   Hence $AC = 2$ and $A(\triangle ABC) = 1$

2. a) Find the smallest positive integer with exactly 100 different factors including 1 and the number itself.

$\tau(n) = 100 = 2 \cdot 2 \cdot 5 \cdot 5$
So smallest possible number will be $2^4 \cdot 3^4 \cdot 5 \cdot 7 = 45360$

3. b) A rectangle can be divided into n equal squares . The same rectangle can be divided into $n + 76$ equal squares . Find n .
Solution:Let the length and breadth of the rectangle be l, b .
Let initially we take square of size axa and $x = \frac{l}{a}$ and $y = \frac{b}{a}$ so $n = xy$
Also Let second time the square be of size cxc so$t = \frac{l}{c}$ and $z = \frac{b}{c}$ so $n + 76 = tz$
Hence $\frac{x}{y} = \frac{l}{b} = \frac{t}{z} = k$
and $76 = tz - xy = kz^2 - ky^2$
As $76 = 2^2 \cdot 19 = k(z - y)(z + y)$
As all k, y, z are integers we get following possibilities (Note for $yz \in N$ we need the parity of $(y + z)$ and $(z - y)$ to be the same)

k	$z - y$	$z + y$	z	y	t	x	$n + 76 = tz$	$n = xy$
1	2	38	20	18	20	18	400	324
4	1	19	10	9	40	36	400	324

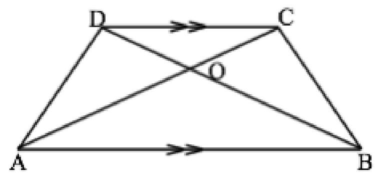
Note in both cases we get $n = 324$.

4. Prove that $1^n + 2^n + \cdots + 15^n$ is divisible by 480 for all odd $n \geq 5$.
Solution: As $a + b \mid a^n + b^n$ for n odd we get
$15 \mid (1^n + 14^n) + (2^n + 13^n) + \cdots + (7^n + 8^n) + 15^n$
Also for $n \geq 5$
$32 \mid (2k)^n$
TPT $32 \mid S = 1^n + 3^n + 5^n + \cdots + 15^n$
Now $S = 1^n + (16-1)^n + 3^n + (16-3)^n + 5^n + (16-5)^n + 7^n + (16-7)^n$
$15^n = (16-1)^{2k+1} = 15\left((16-1)^2\right)^k = 15\left(16^2 - 2 \cdot 16 + 1\right)^k \equiv 15(1)^k (mod 32)$
$\therefore 1 + 15^n = 16 (mod 32)$
$13^n = (16-3)^{2k+1} = 13\left((16-3)^2\right)^k = 13\left(16^2 - 2 \cdot 16 \cdot 3 + 3^2\right)^k \equiv 13(3^2)^k (mod 32)$
$\therefore 3^n + 13^n \equiv 3^{2k+1} + 13 \cdot 3^{2k} = 16 \cdot 3^{2k} (mod 32) \equiv 16 (mod 32)$
Similarly we can show that $5^n + (16-5)^n \equiv 7^n + (16-7)^n \equiv 16 (mod 32)$
$\therefore 32 \mid S$ so $32 \cdot 15 = 480 \mid S$

5. Is it possible to have 19 lines in a plane such that
1) no three lines are concurrent
2) they have exactly 95 points of intersection.
Solution: Let the sets of $a_1, a_2, \cdots, a_k$ lines be parallel so no two lines from the same set will intersect and every two lines from different sets will intersect.
As the the number of lines is 19 so we get $a_1 + a_2 + \cdots + a_k = 19$ and the number of points of intersections will be $\sum a_i a_j = 95$.
$\therefore (a_1 + a_2 + \cdots + a_k)^2 = \left(a_1^2 + a_2^2 + \cdots + a_k^2\right) + 2\left(\sum a_i a_j\right)$

$\therefore \left(a_1^2 + a_2^2 + \cdots + a_k^2\right) = 361 - 2(95) = 171 = 144 + 27 = 144 + 25 + 1 + 1 = 12^2 + 5^2 + 1^2 + 1^2$
And $12 + 5 + 1 + 1 = 19$
So if we have 2 sets of parallel lines with 12 lines in first set and 5 in second set and two lines non parallel to each.

6. In trapezium $ABCD$ with AB parallel to CD , the diagonals intersects at P . The area of $\triangle ABP$ is 72 $cm^2$and $\triangle CDP$ is 50 cm^2 . find the area of the trapezium .



Solution:
Let $O = AC \cap BD$
Note as $AB \parallel DC$ so $\triangle AOB \sim \triangle COD$ so $\frac{AO}{CO} = \frac{BO}{DO} = k$
$\therefore \frac{[\triangle AOD]}{[\triangle COD]} = \frac{[\triangle AOB]}{[\triangle AOD]} = k$
$\therefore [\triangle AOB] = k^2[\triangle COD]$
$\therefore k^2 = \frac{72}{50} = \frac{36}{25} = \left(\frac{6}{5}\right)^2$
$\therefore k = \frac{6}{5}$
$\therefore [\triangle AOD] = [\triangle COB] = \frac{6}{5}(50) = 60$
$\therefore [\square ABCD] = 72 + 2(60) + 50 = 242$

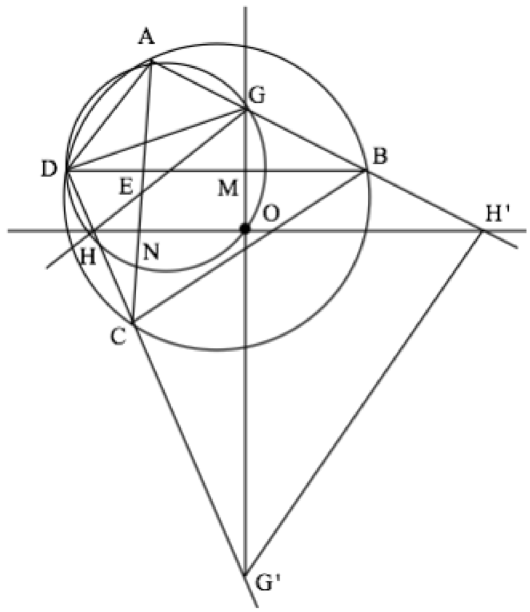
7. Let $a < b < c$ be three positive integers. Prove that among any $2c$consecutive positive integers there exists three different numbers x, y, z such that $abc \mid xyz$.
Proof: Clearly among any $2c$ consecutive integers there will be two divisible by a ,two divisible by b and two divisible by c .
Case 1
If we get distinct x, y, z divisible by a, b, c then $abc \mid xyz$.
Case 2
If there are only two such numbers x, y such that say $a \mid x$ and $b \mid y$ and $c \mid y$, then
Sub-case 1
$gcd(b, c) = 1$ then $bc \mid y$ then for any z we get $abc \mid xyz$
Sub-case 2
$gcd(b, c) = g$ then let $b = gb_0$and $c = gc_0$, $g < c$
Then $gb_0c_0 \mid y$
Also as $g < c$ there exists at least two numbers in given $2c$ numbers divisible by g so there exists a number z such that $g \mid z$ and $z \neq y$
Hence we get $abc \mid xyz$.

8. a) Let m, nbe positive integers. If $m^3 + n^3$is the square of an integer, then prove that $(m + n)$ is not a product of distinct primes .

Solution: Let $m+n=pq$ where $p,q$are distinct primes.
$\therefore m^3+n^3=(m+n)(m^2+n^2-mn)=(m+n)\left((m+n)^2-3mn\right)$
$\therefore pq \mid 3mn$
Case 1
$p\neq 3$ and $q\neq 3$, then $pq \mid mn$
and $pq \mid m+n$ so $pq \mid (m-n)^2=(m+n)^2-2mn$
$\therefore pq \mid m-n$ so $pq \mid 2m$ and $pq \mid 2n$
If $p\neq 2$ and $q\neq 2$ implies $pq \mid m$ and $pq \mid n$ so $pq\leq m$ and $pq\leq n$ so $2pq\leq m+n$ a contradiction .
So WLG let $p=2$ and $q>3$.
Then $q \mid m$ and $q \mid n$ so $pq=2q\leq m+n$ a contradiction.
Case 2
If $p=3$ then$q \mid m$ and $q \mid n$so $m=aq, n=bq$
so $m+n=q(a+b)=3q$ so $(a,b)=(2,1)or(1,2)$
$\therefore m^3+n^3=9q^3$ is not a perfect square.

b) a,b,c are real numbers such that $ab+bc+ca=-1$. Prove that $a^2+5b^2+8c^2\geq 4$.
Proof: $a^2+5b^2+8c^2=a^2+b^2+4b^2+4c^2+4c^2$ so we need to try combinations that has these square terms and the middle terms have **the same coefficients** so that the middle term is of the form $k(ab+bc+ca)$
By trail and error we get such combinations
1) $(a+2b+2c)^2+(b-2c)^2=\left(a^2+4b^2+4c^2+4ab+4ac+8bc\right)+\left(b^2+4c^2-4bc\right)$
$=a^2+5b^2+8c^2+4(ab+bc+ca)=a^2+5b^2+8c^2-4\geq 0$
$\therefore a^2+5b^2+8c^2\geq 4$

9. $ABCD$ is a quadrilateral in a circle whose diagonals intersect at right angles . Through O the center of the circle GOG' and HOH' are drawn parallel to AC,BD respectively , meeting AB, CDin G, H and DC, AB produced at G', H' . Prove that $GH, G'H'$ are parallel to BC and AD respectively.



Proof:Let $M = OG \cap BD$ and $N = AC \cap HO$

As perpendicular from center bisect the chord so OG bisects BD and OH bisects AC.

$\therefore \triangle GBM \cong \triangle GDM$ and $\triangle HAN \cong \triangle HCN$ so

Let $\angle GBM = \angle GDM = y$

$\therefore \angle ACH = \angle HAC = y$

Let $\angle HGM = x, \angle DGH = a$ so we get $\angle DGM = \angle BGM = \angle AHN = \angle CHN = x + a$

$\therefore y + x + a = 90 - - - - - - -(1)$

$\therefore$ In $\square DGOH$, $\angle DGM = \angle OHC = x + a$ making it cyclic.

Also in $\square AHOG$,$\angle AHO = \angle BGO = x + a$ making it cyclic.

$\therefore$ Points A, G, O, H, D are concyclic.

$\therefore \angle DAH = \angle DGH = a$

$\therefore \angle DBC = \angle DAC = y + a$

$\therefore \angle ABC = y + y + a = 2(90 - x - a) + a = 180 - 2x - a$ from(1)

Also $\angle AGH = \angle AGD + \angle DGH = \angle AHD + a = 180 - 2(x + a) + a = 180 - 2x - a$

$\therefore \angle ABC = \angle AGH$ making $GH \parallel BC$. By similar argument it can be shown that $G'H' \parallel AD$