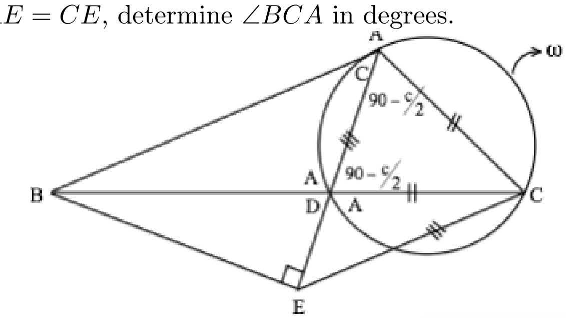
IIT & MEDICAL

INMO2019

January 22, 2019

1. Let ABC be a triangle with $\angle BAC > 90$. Let D be a point on BC and E be a point on the AD such tat AB is tangent to the circumcircle of triangle ACD at $A$and BE is perpendicular to AD . Given that $CA = CD$ and $AE = CE$, determine $\angle BCA$ in degrees.



Solution: Consider the power of point of B wrt ω , $AB^2 = BD \cdot BC$

$\therefore BD = \frac{c^2}{a}$ and $CD = a - \frac{c^2}{a} = \frac{a^2-c^2}{a} = CA = b$ (Given)

$\therefore \boxed{a^2 - c^2 = ab} - - - - - - (1)$

Note be tangent secant $\angle BAD = \angle C$ so in ΔABD, $\angle ADB = \angle BAC = A$(say)

$\therefore A = 90 + \frac{C}{2}$

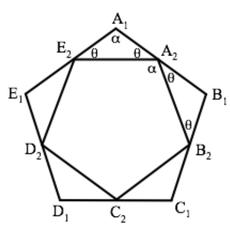
$\therefore$ In ΔABC by sine rule

$\frac{a}{c} = \frac{\sin A}{\sin C} = \frac{\sin\left(90+\frac{C}{2}\right)}{\sin C} = \frac{\cos\frac{C}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{1}{2\sin\frac{C}{2}}$

$\therefore \boxed{\frac{a^2}{c^2} = \frac{1}{4\sin^2\frac{C}{2}}} - - - - - (2)$

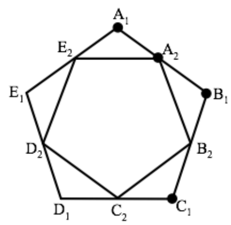
In ΔBEA, $AE = c\ \cos C - - - - - (3)$
In $\Delta AEC, AE = CE$
$\therefore \angle EAC = \angle ECA = 90 - \frac{C}{2}$ so $\angle AEC = C$
In ΔCED , $\frac{CE}{\sin A} = \frac{CD}{\sin C} = \frac{CA}{\sin C}$ so $CE = \frac{b \sin A}{\sin C} = b\left(\frac{a}{c}\right)$
As $AE = CE$
From (3) we get $c\ \cos C = \frac{ab}{c}$
From (1) we get $c^2\ \cos C = a^2 - c^2$
$\therefore c^2(1 + cosC) = a^2$
$\therefore 2\ \cos^2\frac{C}{2} = \frac{a^2}{c^2} = \frac{1}{4\sin^2\frac{C}{2}}$
$\therefore 4\ \cos^2\frac{C}{2}\ \sin^2\frac{C}{2} = \frac{1}{2}$
$\left(2\ \cos\frac{C}{2}\ \sin\frac{C}{2}\right) = \sin C = \pm\frac{1}{\sqrt{2}}$
As C is acute angle of a triangle so $\sin C = \frac{1}{\sqrt{2}}$ so $\boxed{\angle C = 45^\circ}$

2. Let $A_1B_1C_1D_1E_1$be a regular pentagon. For $2 \le n \le 11$,let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoints of the sides of $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}$. All 5 vertices of each of 11 pentagons are arbitrarily coloured red or blue. Prove that 4 points among these 55 points have the same colour and form the vertices of a cyclic quadrilateral.
Solution1
First we will prove that $A_iB_iC_iD_iE_i$ is regular by induction.
Let the angles of pentagons be α each and let $\angle A_1E_2A_2 = \angle A_1A_2E_2 = \angle B_1A_2B_2 = \theta$ so $\alpha + 2\theta = \pi$
Hence $\angle E_2A_2B_2 = \alpha$.
So all angles of $A_2B_2C_2D_2E_2$are α each .
Also by midpoint theorem $A_2B_2 = \frac{1}{2}A_1C_1$. so all sides are equal giving $A_2B_2C_2D_2E_2$ regular .

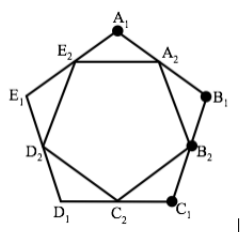


By induction it can be proved all pentagons are$A_iB_iC_iD_iE_i$ regular.
If any 4 vertices of a pentagon are of the same colour we get a cyclic monochromatic quadrilateral.
So, we must have 2 vertices of one color and 3 vertices of the other. Let $V = \{A, B, C, D, E\}$, note that any $X_iX_jY_iY_j$ is an isosceles trapezium where $X, Y \in V$, $X \neq Y$, $i \neq j$, $1 \le i, j \le 11$.
So, it suffices to prove that there exists a monochromatic isosceles trapezium $X_iX_jY_iY_j$.
Consider a 5×11 matrix filled with 1's and 0's, as per the colour of it's vertices.
The monochromatic isosceles trapezium $X_iX_jY_iY_j$ will represent a rect-

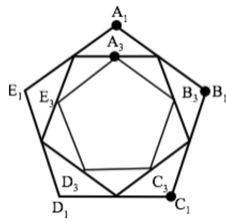
angle on matrix with all one's or all zero's.
Since each row has at least 2 *one*'s and there are at most $10 = \binom{5}{2}$ ways to place two reds, so by PHP, one pair of position must be repeated in 11 rows so we get such a rectangle.
Second solution: Clearly in any pentagon if 4 or more vertices are of the same colour, we get cyclic quadrilateral. So we assume that each pentagon has either $(3R + 2B)$ or $(3B + 2R)$ vertices. WLG let $A_1B_1C_1D_1E_1$ has $3R + 2B$ vertices.
Case I: $3R$ vertices are adjacent say $A_1B_1C_1$



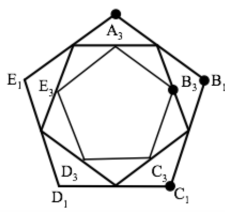
Sub case I: $A_2B_2C_2D_2E_2$ has **3 red** vertices.
1) If A_2 is R
If B_2 is R then $\Box A_1A_2B_2C_1$ is an isosceles trapezium.
If C_2 is R then $\Box A_1A_2C_1C_2$ is an isosceles trapezium.
If B_2, C_2 are B then D_2, E_2 are$R$ then $\Box B_1E_2D_2C_1$ is an isosceles trapezium.



2) If $A_2$is B and B_2 is R
If E_2 is R then $\Box E_2A_1B_1B_2$ is an isosceles trapezium.
If E_2 is B then D_2, C_2 are$R$ then $\Box A_1B_1C_2D_2$ is an isosceles trapezium.
Sub case II: $A_2B_2C_2D_2E_2$ has **2 red** vertices.
Consider $A_3B_3C_3D_3E_3$ has **2red** and **3blue** , then by previous argument we get cyclic quadrilateral among the vertices of $A_2B_2C_2D_2E_2$ and$A_3B_3C_3D_3E_3$.
If $A_3B_3C_3D_3E_3$ has **3red** and **2blue**



1) If A_3 is R
If B_3 is R then $\square A_1B_1B_3A_3$ is an isosceles trapezium.
If C_3 is R then $\square A_1C_1C_3A_3$ is an isosceles trapezium.
If B_3, C_3 are B then D_3, E_3 are$R$ then $\square A_1C_1D_3E_3$ is an isosceles trapezium.



2) If $A_3$is B and B_3 is R
If C_3 is R then $\square B_3B_1C_3C_1$ is an isosceles trapezium.
If C_3 is B thenD_3E_3 are$R$ then $\square E_3A_1B_1D_3$ is an isosceles trapezium.
Case II: $3R$ **vertices are not adjacent say** $A_1B_1D_1$
Sub case I: $A_2B_2C_2D_2E_2$ has **3 red** vertices.
1) If A_2 is R
If B_2E_2 is R then $\square A_1E_2B_2B_1$ is an isosceles trapezium.
If B_2D_2 is R then $\square A_1E_2D_2D_1$ is an isosceles trapezium.
If B_2, C_2 are R then $\square A_2B_2C_2D_1$ is an isosceles trapezium.
If E_2, C_2 are R then $\square A_2C_2D_1A_1$ is an isosceles trapezium.
2) If $A_2$is B and B_2 is R
Either B_2C_2 is R then $\square A_1B_2C_2D_1$ is an isosceles trapezium.
or E_2D_2 is R then then $\square A_1E_2D_2D_1$ is an isosceles trapezium.
Sub case II: $A_2B_2C_2D_2E_2$ has **2 red** vertices.
Consider $A_3B_3C_3D_3E_3$ has **2red** and **3blue** , then by previous argument we get cyclic quadrilateral among the vertices of $A_2B_2C_2D_2E_2$ and$A_3B_3C_3D_3E_3$.
If $A_3B_3C_3D_3E_3$ has **3red** and **2blue**
1) If A_3 is R
If B_3 is R then $\square A_1B_1B_3A_3$ is an isosceles trapezium.
If D_3 is R then $\square A_1A_3D_3D_1$ is an isosceles trapezium.
If B_3, D_3 are B then C_3, E_3 are$R$ then $\square A_1B_1C_3E_3$ is an isosceles trapezium.
2) If $A_3$is B and B_3 is R
If D_3 is R then $\square B_1B_3D_3D_1$ is an isosceles trapezium.

If D_3 is B then E_3, C_3 are$R$ then $\square A_1E_3C_3B_1$ is an isosceles trapezium.

3. Let m, n be distinct positive integers . Prove that

$$gcd(m,n) + gcd(m+1, n+1) + gcd(m+2, n+2) \leq 2 \mid m-n \mid +1$$

Further determine when equality holds.
Solution:
As $gcd(m,n) = gcd(m-n, n)$
$gcd(m+1, n+1) = gcd(m-n, n+1)$
$gcd(m+2, n+2) = gcd(m-n, n+2)$
Maximum value of these can be $m-n$.
But exactly one of them can be $m-n$, as no prime number can divide consecutive integers.
Then other two are the factors of $(m-n)$ so can be at most $\frac{1}{2}(m-n)$
Hence for $\mid m-n \mid \geq 3$ we have
$gcd(m-n, n) + gcd(m-n, n+1) + gcd(m-n, n+2) \leq 2 \mid m-n \mid$
1) For $\mid m-n \mid = 1$
LHS=3 RHS=3 we get equality in $gcd(m-n, n) + gcd(m-n, n+1) + gcd(m-n, n+2) \leq 2 \mid m-n \mid +1$
2) For $\mid m-n \mid = 2$, at most two of $n, (n+1), (n+2)$ are even so atmost two of the gcds are 2
LHS=5 RHS=5 we get equality when m, n are even .

4. Let $n$and Mbe positive integers such that $M > n^{n-1}$. Prove that there are distinct primes $p_1, p_2, \cdots, p_n$such that p_j divides $M + j$ for $1 \leq j \leq n$.
Solution:
Hall's Marriage Theorem:
Let G be a bipartite graph with bipartite sets $X$and Y . Then there exists a matching that covers Xiff for each subset W of X

$$\mid W \mid \leq \mid N(W) \mid$$

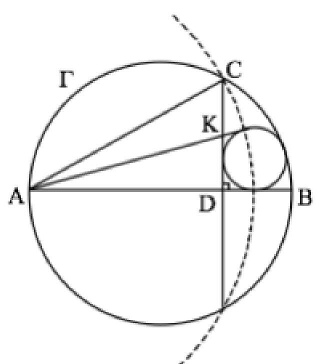
where $N(W)$ is the set of neighbours of W in Y that is $N(W) = \{v \in Y/vu$ is an edge in G for some $u \in W$
We wish to prove that we have distinct primes available for each of the number.
We wish to prove that for any subset A of $\{M+1, M+2, \cdots, M+n\}$ the product of it's elements has more factor than $\mid A \mid$ .So it satisfy the condition for Hall's marriage theorem. So we can match one one factor with each number.
If $(M+j)$ has at least n factors then it satisfies the condition.
So let us consider the case when the number of prime factors of $(M+j)$ is less than or equal to $n-1$.
Let $M + j = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ with $k \leq n-1$.
Let $f(M+j)$ =Max$\{p_1^{a_1}, p_2^{a_2}, \cdots, p_k^{a_k}\}$.
We will pair $f(M+j)$ with the element $(M+j)$.

If bases of all $f(M+j)$ are distinct we are done.
If not then there exists index j such that $p_j^{a_j} = f(M+a)$and $p_j^{b_j} = f(M+b)$.

$$n < \sqrt[n-1]{m} < \min\left(\sqrt[n-1]{M+a},\ \sqrt[n-1]{M+b}\right) ----(1)$$

Now $M+a = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$
As$p_j^{a_j} \geq p_1^{a_1}$and $p_j^{a_j} \geq p_2^{a_2}\cdots$ and $p_j^{a_j} \geq p_k^{a_k}$
$\therefore \left(p_j^{a_j}\right)^k \geq p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$
As $k \leq n-1$ so $\left(p_j^{a_j}\right)^{n-1} \geq p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k} = M+a$
$\therefore \left(p_j^{a_j}\right) \geq \sqrt[n-1]{M+a}$ where $a_j = v_p(M+a)$
and $\therefore \left(p_j^{b_j}\right) \geq \sqrt[n-1]{M+a}$ where $b_j = v_p(M+b)$
$\therefore$From (1) $n < min\left\{\left(p_j^{a_j}\right), \left(p_j^{b_j}\right)\right\} = p_j^{min\{a_j,b_j\}} ----- (2)$
Now $p_j^{a_j} \mid M+a$ and $p_j^{b_j} \mid M+b$
$\therefore p_j^{min\{a_j,b_j\}} \mid (\mid (M+a)-(M+b) \mid) < n ----- (3)$
As (2) and (3) implies contradiction , hence there do note exists such a index . So every $(M+j)$ has distinct prime divisor .

5. Let AB be a diameter of a circle Γ and let Cbe a point on Γ different from A and B . Let D be the feet of perpendicular the from C on to AB. Let K be a point of the segment CD such that AC is equal to the semiperimeter of the $\triangle ADK$. Show that the excircle of $\triangle ADK$ opposite A is tangent to Γ.



Solution:
From $\triangle ADC \sim \triangle ACB$ we get $AC^2 = AD \cdot AB$
Consider inversion about A in radius AC. This sends $B \to D$.
It can be seen that this sends circumcircle w of ΔABC to the line CD.
$AC =$ semiperimeter of $\Delta ADK =$ tangent length from A to A-excircle of ΔADK
$\therefore$ The A-excircle of ΔADK is orthogonal to the circle of inversion and thus remains invariant. Since the A-excircle is tangent to line CD, we see

that the image of the A-excircle (which is itself) is tangent to the image of line CD, which it the circle w.
$\therefore$ The A-excircle of ΔADK is tangent .

6. Let f be the function defined from the set $\{(x,y) : x,y \in R, xy \neq 0\}$ to the set of all positive real numbers such that
a) $f(xy,z) = f(x,z)f(y,z)$ for all $x,y \neq 0$
b) $f(x,1-x) = 1$ for all $x \neq 0,1$
Prove that
a) $f(x,x) = f(x,-x) = 1$ for all $x \neq 0$
b) $f(x,y)f(y,x) = 1$ for all $x,y \neq 0$
THIS PROBLEM IS INCORRECT
This problem, however, is incorrect as the following function serves as a counterexample and satisfies the premises of the questions but not the statements to be proven:

$$f(x,y) = \begin{cases} \mid x \mid & \text{if } y = 2 \\ 1 & \text{otherwise} \end{cases}$$

As we can see that this function satisfies (i) as if $z = 2$ then $\mid xy \mid = \mid x \mid \cdot \mid y \mid$
and if $z \neq 2$ then both $LHS = RHS = 1$
Also to satisfy the condition (ii), if $1 - x = 2$, that is $x = -1$ then
$f(x, 1-x) = \mid x \mid = 1$
and if $x \neq -1$, then, $1 - x \neq 2$, and by definition $f(x, 1-x) = 1$
but
$f(2,2) = 2$ which contradicts the condition (a) $f(x,x) = f(x,-x) = 1$ for all $x \neq 0$ that was to be proved.