IIT – JEE: 2020 **OLYMPIAD TEST** **DATE: 17/08/18**

PHYSICS

SOLUTION

1. **(C)**

Average acceleration,

$$a = \frac{v_f - v_i}{\Delta t} = \frac{\tan\theta_2 - \tan\theta_1}{20} = 0 \qquad (\because \theta_2 = \theta_1)$$

2. **(B)**

Resistance $= kv\left(= \frac{ds}{dt}\right)$

Equations of motion are

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \qquad ......(1)$$

$$\frac{d^2y}{dt^2} = -k\frac{dy}{dt} - g \qquad ......(2)$$

Integrating (1) and (2) and using the initial conditions,
We get

$$\frac{dx}{dt} = u\cos\alpha . e^{-kt} \qquad ......(3)$$

And $k\frac{dy}{dt} + g = (ku\sin\alpha + g).e^{-kt}$

i.e., $\frac{dy}{dx} = \frac{1}{k}\left[(ku\sin\alpha + g).e^{-kt} - g\right] \qquad ......(4)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left[(ku\sin\alpha + g).e^{-kt} - g\right]}{ku\cos\alpha . e^{-kt}} \qquad ......(5)$$

Direction of projection was α with the horizontal, when the direction of motion again makes the angle α with the horizontal, it really makes the angle $(\pi - \alpha)$ with the horizontal in the sense of the direction of projection. If this happens after the time *t*, we have from (5),

$$\tan(\pi - \alpha) = \frac{(ku\sin\alpha + g).e^{-kt} - g}{ku\cos\alpha . e^{-kt}}$$

i.e., $-\tan\alpha = \frac{(ku\sin\alpha + g) - ge^{-kt}}{ku\cos\alpha}$

i.e., $-ku\sin\alpha = ku\sin\alpha + g - g.e^{-kt}$

or $e^{kt} = 1 + \frac{2ku}{g}\sin\alpha$

or $t = \frac{1}{k}\log\left(1 + \frac{2ku}{g}\sin\alpha\right)$

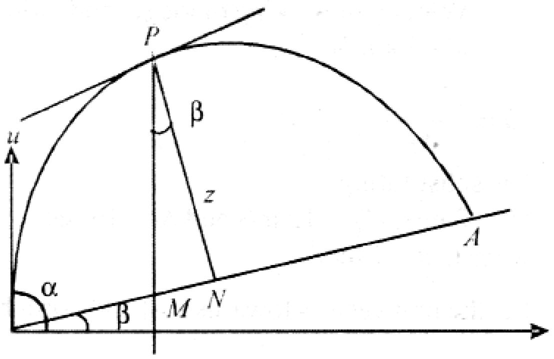
3. **(D)**

This is a straight forward problem in calculating the total distance. The only trick is to make sure that in the cross-over lap you use the hypotenuse. The total distance must be precisely 400 m, so we must have $400 = 112 + \pi(R+0.05) + \pi(R+5.05) + (112^2 + 5^2)^{1/2}$.

Solving for R gives 25.44 m.

4. **(B)**

P be the point where the tangent is parallel to the inclined plane. If $PN = z$ be perpendicular from P on the inclined plane and PM the vertical altitude of P then evidently for all points on the path, P is the point where z is the greatest and consequently PM is the greatest.



Now for the point P, velocity perpendicular to the inclined plane is zero. Now the velocity and acceleration perpendicular to the plane at O is $u\sin(\alpha-\beta)$ and $g\cos\beta$ and this velocity becomes zero at P.

$\therefore\ 0 = u^2\sin^2(\alpha-\beta) - 2g\cos\beta . z$

$$z = \frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta}$$

For maximum range $\alpha = \frac{\pi}{4} + \frac{\beta}{2}$ or $\alpha - \beta = \frac{\pi}{4} - \frac{\beta}{2}$

Hence, $z = \frac{u^2}{2g\cos\beta}\sin^2\left(\frac{\pi}{4} - \frac{\beta}{2}\right)$

$$= \frac{u^2}{4g\cos\beta}\left[1 - \cos\left(\frac{\pi}{2} - \beta\right)\right] =$$

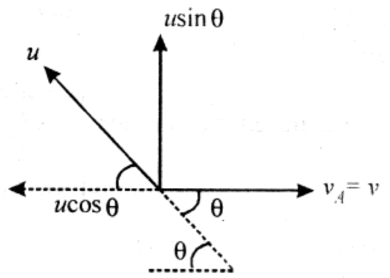
$$= \frac{u^2}{4g\cos\beta}(1 - \sin\beta) \text{ or } PM = z\sec\beta$$

$$= \frac{u^2}{4g\cos^2\beta}(1-\sin\beta) = \frac{u^2}{4g(1+\sin\beta)} = \frac{1}{4} \text{ (maximum range)}$$

$\Rightarrow$ Maximum range $= 4 \times PM$

5. **(B)**

$\vec{v}_B = \vec{v}_{BA} + \vec{v}_A$



$u\cos\theta = v \Rightarrow \theta = \cos^{-1}\left(\frac{v}{u}\right)$

6. **(A)**

$\vec{v}_{SB} = \upsilon\hat{j} = \vec{v}_S + 3\hat{i}$

$\therefore \; v_S = v\hat{j} - 3\hat{i}$ and $\upsilon = \frac{100}{50} = 2 \text{ m/s}$

$\therefore \; |\vec{v}_S| = \sqrt{\upsilon^2 + 3^2} = \sqrt{13}\,\text{m/s}$

Drift = 50 × 3 = 150 m

7. **(B)**

The component of acceleration due to gravity along greatest slope is $g' = g\sin 30^\circ = 10 \times \frac{1}{2} = 5\,\text{ms}^{-2}$

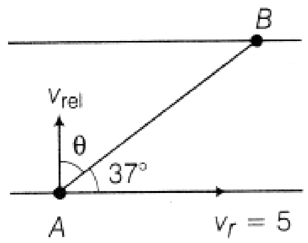
$\therefore \; R = \frac{u^2 \sin 2\alpha}{g}$ or $120 = \frac{u^2 \sin 120^\circ}{5}$

$\Rightarrow \; u^2 = \frac{120 \times 5 \times 2}{\sqrt{3}}$

$\therefore \; u = \sqrt{\frac{1200}{1.732}} = 26.3\,\text{ms}^{-1}$

8. **(B)**

Here, $v_{\text{rel}} = \sin\theta = 5\sin 37^\circ$



$\Rightarrow \; v_{\text{rel}} \sin\theta = 5 \times \frac{3}{5} = 3$

$\Rightarrow \; v_{rel} \sin\theta = 3 \qquad \ldots(\text{i})$

Also, $t = \frac{AB}{v_r \cos 37^\circ + v_{\text{rel}} \cos\theta}$

$\Rightarrow \; 2 = \frac{8 + 6\sqrt{3}}{5 \times \frac{4}{5} + v_{\text{rel}} \cos\theta}$

$\Rightarrow \; 4 + v_{\text{rel}} \cos\theta = 4 + 3\sqrt{3}$

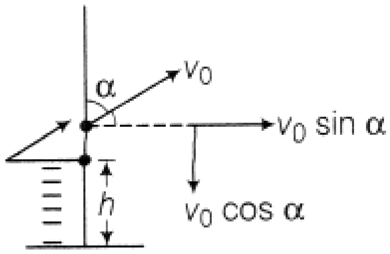
$\Rightarrow \; v_{rel} \cos\theta = 3\sqrt{3}$

$\therefore \; (v_{rel}\cos\theta)^2 + (v_{rel}\sin\theta)^2 = 3^2 + (3\sqrt{3})^2$

$\therefore \; v_{rel} = 6\ ms^{-1}$

9. **(B)**

Time taken to reach ground $= 10t_0$



$\Rightarrow \; h = v_0 \sin\alpha (10t_0) - \frac{1}{2} g (10t_0)^2$

$h = v_0 \sin\alpha (10t_0) - 50 g t_0^2$

$\Rightarrow \; 50 g t_0^2 - 10 v_0 (\sin\alpha) t_0 + h = 0 \qquad ...(i)$

Time taken by boy $= 5t_0$

$h = \frac{1}{2} g (5t_0)^2 \Rightarrow t_0 = \sqrt{\frac{2h}{25h}} \qquad ...(ii)$

For the time $t_0 < t < 5t_0$, boy and ball fall with same acceleration (g downwards).

$\Rightarrow \; a_{rel} = 0$

They see each other performing straight line motion.

For time $5t_0 < t < 10t_0$, boy will remain stationary on ground and ball will come down with acceleration g.

$\therefore$ It will again trace a parabolic path sort boy.

Also, we can get b from Eq. (ii) from given t_0 and ca thus also find v_0 using equation.

(i) As till $t <$ boy is still on building's roof hell sec ball's trajectory as

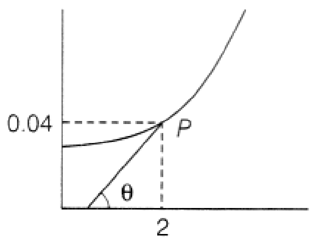
$y = v_0 \sin\alpha t - \frac{1}{2} g t^2$

$\Rightarrow \; x = v_0 \cos\alpha t$

$\Rightarrow \; t = \frac{x}{v_0 \cos\alpha}$

$\Rightarrow \; y = x \tan\alpha - \frac{g x^2}{2u^2 \cos^2\alpha}$

10. **(C)**



From figure, slope is $\tan\theta = \frac{0.04}{2}$ or $\frac{1}{t_0 v_0} = \frac{0.04}{2}$

$\therefore \quad v_0 t_0 = \frac{2}{0.04} = \frac{200}{4} = 50 \text{ mm}$

11. **(ABCD)**

Change in velocity = final velocity – initial velocity

$= u\cos\theta\hat{i} - \left(u\cos\theta\hat{i} + u\sin\theta\hat{j}\right) = -u\sin\theta\hat{j}$

$\therefore$ (A) is correct

Average velocity = (total displacement)/(time taken) = ($R\hat{i}$ / Time of flight)

$= u\cos\theta\hat{i}$

$\therefore$ (B) is correct.

Change in velocity = final velocity – initial velocity

$= \left(u\cos\theta\hat{i} - u\sin\theta\hat{j}\right) - \left(u\cos\theta\hat{i} + u\sin\theta\hat{j}\right)$

$= -2u\sin\theta\hat{j}$

$\therefore$ (C) is also correct.

Rate of change of momentum = force

Constant gravitational force is acting on the projectile.

$\therefore$ (D) is also correct.

12. **(AB)**

Equation of motion is

$\frac{dv}{dt} = -\mu v^3$ i.e. $-\frac{dv}{v^3} = \mu\, dt$

Integrating it, we have $\frac{1}{v^2} = 2\mu t + A$ .....(1)

Initially, when $t = 0, v = V$

$\therefore A = \frac{1}{V^2}$

Hence, equation (1) becomes

$\frac{1}{v^2} = 2\mu t + \frac{1}{V^2} = \frac{1 + 2\mu V^2 t}{V^2}$

i.e. $v = \frac{dx}{dt} = \frac{V}{\sqrt{\left(1 + 2\mu V^2 t\right)}}$

This proves the second option.

Integrating it, we have

$x = V\int\left(1 + 2\mu V^2 t\right)^{-1/2} dt + D$

$= \frac{1}{\mu V}\sqrt{\left(1 + 2\mu V^2 t\right)} + D$

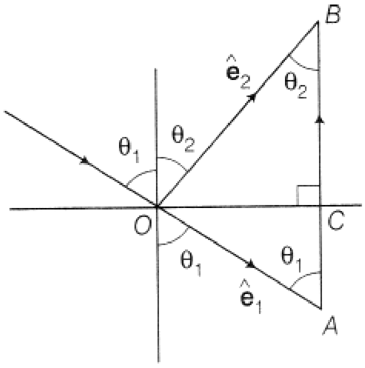
Initially when $t = 0, x = 0$

$\therefore D = -\frac{1}{\mu V}$

So that, we have $x = \frac{1}{\mu V}\left[\sqrt{\left(1 + 2\mu V^2 t\right)} - 1\right]$

13. **(ABC)**

Here, $OB = OA + AB$



or $\quad \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_1 + AB\hat{\mathbf{n}} = \hat{\mathbf{e}}_1 + 2AC\hat{\mathbf{n}} \qquad \begin{pmatrix} \because \ \theta_1 = \theta_2 \\ \therefore \ AC = CB \end{pmatrix}$

$= \hat{\mathbf{e}}_1 + 2OA\cos\theta_1\hat{\mathbf{n}}$

$= \hat{\mathbf{e}}_1 + 2|\hat{\mathbf{e}}_1|\cos\theta_1\hat{\mathbf{n}} \qquad [\because \ OA = |\hat{\mathbf{e}}|]$

$= \hat{\mathbf{e}}_1 + 2\cos\theta_1\hat{\mathbf{n}}$

But $\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}} = |\hat{\mathbf{e}}_1||\hat{\mathbf{n}}|\cos(\pi - \theta_1) = -\cos\theta_1$

$\therefore \quad \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_1 - 2(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \qquad .....(i)$

(B) $\because \ -\cos\theta_1 = \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}}$

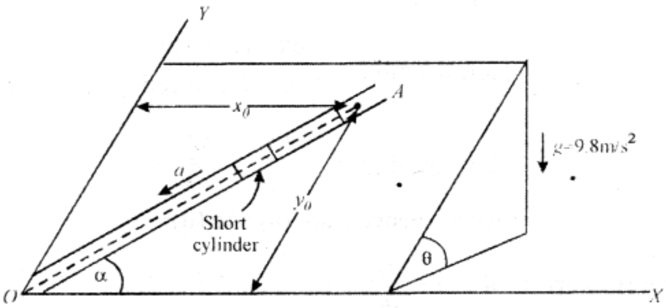
$\Rightarrow \quad \cos\theta_1 = (\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{n}})$

(C) Similarly, $\cos\theta_2 = |\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{n}}|$

14. **(ABC)**

(A) The downward component of g parallel to OY is $g \sin\theta$,

Hence the downward component along the groove is $a = g \sin\theta \sin\alpha$. Since



$\sin\alpha = \dfrac{y_0}{(x_0^2 + y_0^2)^{1/2}} = 0.8$ and $\sin\theta = 0.5$

$a = (9.8)(0.5)(0.8) = 3.92 \text{ m/s}^2$

(B) $s = v_0 t + \dfrac{1}{2}at^2$, where

$s = (x_0^2 + y_0^2)^{1/2} = 5m$ and $v_0 = 0$.

Thus, $s = \dfrac{1}{2}(3.92)t^2$ or $t = 1.597\text{s}$

(C) $v = 0 + (3.92)(1.597) = 6.26 \text{ m/s}$

15. (ABCD)

(A) $\frac{dy}{dt} = -bke^{-bt} + \frac{g}{b}$

Differentiating once more $\frac{d^2y}{dt^2} = b^2ke^{-bt}$

Multiplying our expression for $\frac{dy}{dt}$ by $-b$ and adding g yields b^2ke^{-bt}.

Thus (1) is satisfied. Substituting $t = 0$ and recalling that $e^0 = 1$, we get $y = 0$.

(B) Since the ball is released from rest, $y = 0$ at $t = 0$.

Using our expression for $\frac{dy}{dt}$ from (A),

We have $0 = -bk + \frac{g}{b}$, which yields $k = \frac{g}{b^2}$.

(C) If $b = 0.1/\text{s}$ and using $g = 9.8\ \text{m/s}^2$,

We have $k = \frac{g}{b^2} = 980\,\text{m}$. Then at $t = 10\,\text{s}$, $y = (980\,\text{m})(e^{-1} - 1) + (98\,\text{m/s})(10\,\text{s}) = 360\,\text{m}$.

$v = \frac{dy}{dt} = (-98\,\text{m/s})e^{-1} + (9.8\,\text{m/s}^2)(0.1/\text{s}) = 62\ \text{m/s}$

(D) At $t = 60\,\text{s}$,

$\frac{dy}{dt} = (-98\,\text{m/s})e^{-6} + 98\,\text{m/s}$

$e^{-6} \approx 0.0025$

So, $\frac{dy}{dt} \approx 9.8\,\text{m/s}$

16. (1)

$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 28.56^\circ$

Let $v_0 =$ initial velocity of auto

$s_x = v_{0x} \times t$

$40 = v_0 \cos 28.56^\circ \times t$

$t = \frac{45.54}{v_0}$ .....(1)

Also $s_y = v_{0y}t - \frac{1}{2}gt^2 = v_0 \sin 28.56^\circ \times t - \frac{1}{2}gt^2$

$s_y = v_0 \sin 28.5^\circ \times \frac{45.54}{v_0} - \frac{1}{2} \times 32.2 \times \left(\frac{45.54^2}{v_0}\right)$

$= -10$ (given)

Solving this, $v_0 = 32.76$ ft/sec.

$v_{0x} = 32.76 \cos 28.56^\circ = 29.3\ \text{ft/sec}$

$v_{0y} = 32.76 \sin 28.56^\circ = 15.66\ \text{ft/sec}$

Also $v_y^2 = v_{0y}^2 + 2a_y s_y = (15.66)^2 - 2(32.2)(-10)$

$$v_y = 29.3 \text{ ft/s}, v_x = 29.3 \text{ ft/s}; \ \alpha = \tan^{-1}\frac{v_y}{v_x} = 45^\circ$$

17. **(4)**

$$\frac{s\upsilon_p}{s+\upsilon_p(\Delta t_1 - \Delta t_2)} = 40 \text{ km/h}$$

18. **(3)**

$$n + \left(\frac{\Delta t_2^2 - \Delta t_1^2}{\Delta t_1^2}\right) = 18$$

19. **(8)**

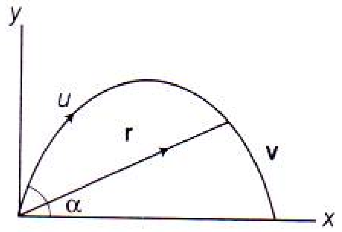
The position vector is $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$

The ball is at the greatest distance from the origin when its velocity is perpendicular to position vector.

$$\therefore \quad \frac{y}{x} = \frac{-v_y}{v_x}$$

$$\frac{ut\sin\alpha - \frac{1}{2}gt^2}{ut\cos\alpha} = -\frac{(u\sin\alpha - gt)}{u\cos\alpha}$$

$$t^2 - \frac{3u\sin\alpha}{g}t + \frac{2u^2}{g^2} = 0$$



If this is not happen during motion, the discriminant should be negative.

$$\therefore \quad \left(\frac{3u\sin\alpha}{g}\right)^2 < \frac{8u^2}{g^2}$$

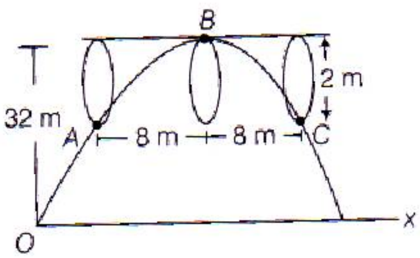
$$\Rightarrow \quad \sin\alpha < \frac{\sqrt{8}}{3}$$

$$\therefore \quad n = 8$$

20. **(1)**

In figure, points *A*, *B* and *C* are lying on the trajectory of the path of the ball.

The equation of trajectory is



$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

For point A, $32 - 2 = x\tan\alpha - \dfrac{5x^2}{u^2\cos^2\alpha}$

Or $\quad 30 = x\tan\alpha - \dfrac{5}{u^2}x^2\left(1+\tan^2\alpha\right) \qquad \ldots(\text{i})$

For point B, $32 = \left(x+8\right)\tan\alpha - \dfrac{5}{u^2}\left(x+8\right)^2\left(1+\tan^2\alpha\right) \qquad \ldots(\text{ii})$

For point C, $32 - 2 = \left(x+16\right)\tan\alpha - \dfrac{5\left(x+16\right)^2}{u^2}\left(1+\tan^2\alpha\right)$

$\Rightarrow \quad 30 = \left(x+16\right)\tan\alpha - \dfrac{5\left(x+16\right)^2\left(1+\tan^2\alpha\right)}{u^2} \qquad \ldots(\text{iii})$

Subtracting Eq. (i) from Eq. (ii),

$$2 = 8\tan\alpha - \frac{5}{u^2}\left(1+\tan^2\alpha\right)\left(x^2+16x+64-x^2\right)$$

$\Rightarrow \quad 2 = 8\tan\alpha - \dfrac{5}{u^2}\left(1+\tan^2\alpha\right)\left(16x+64\right) \qquad \ldots(\text{iv})$

Subtracting Eq. (i) from Eq. (iii),

$$0 = 16\tan\alpha - \frac{5\left(1+\tan^2\alpha\right)}{u^2}\left(x^2+32x+256-x^2\right) \qquad \ldots(\text{v})$$

From Eq. (iv) and Eq. (v),

$$\tan^2\alpha = \frac{32\times 2}{64} = 1$$

$\therefore \quad \tan\alpha = 1$