IIT – JEE - 2017 **TW TEST [BATCH – 1 & 5]**

TOPIC: STRAIGHT LINES AND PAIR OF LINES **DATE: 19/10/15**

[SOLUTION]

1. Since x_1, x_2, x_3 and y_1, y_2, y_3 are in G. P. with same common ratio, therefore

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_3}{x_2 - x_3}$$

Hence, Points A_1, A_2, A_3 are collinear,

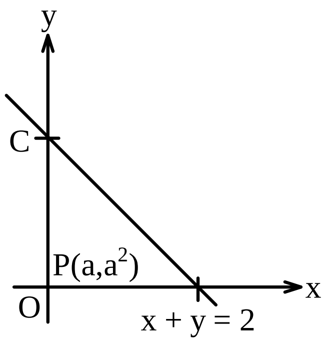
2. Clearly, $a \in R^+$

Also, $a^2 + a - 2 < 0$

$\Rightarrow$ $(a+2)(a-1) < 0$

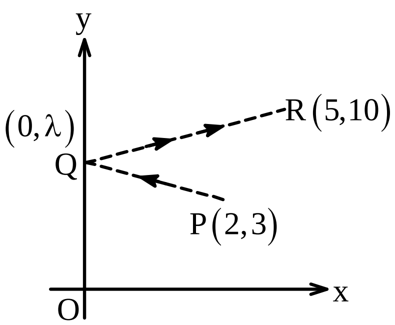
$\Rightarrow$ $-2 < a < 1$

$\Rightarrow$ $a \in (0,1)$



3. If P_1 be the reflection of P in y – axis, then

$P_1 \equiv (-2,3)$



Equation of line P_1R is

$\Rightarrow$ $(y-3) = \frac{10-3}{5+2}(x+2)$

$\Rightarrow$ $y = x + 5$

It meets y – axis at $(0,5)$

$\Rightarrow$ $Q \equiv (0,5)$

4. Lines $x\cos\alpha + y\sin\alpha = p$ and

$x\sin\alpha - y \qquad \cos\alpha = 0$

Are mutually perpendicular.

Thus, $ax + by + p = 0$ will be equally inclined to these lines and would be the angle bisector of these lines.

Now equation of angle bisectors is,

$$x\sin\alpha - y\cos\alpha = \pm(x\cos\alpha + y\sin\alpha - p)$$

$$\Rightarrow x(\cos\alpha - \sin\alpha) + y(\sin\alpha\,\alpha + \cos\alpha) = p$$

Or $x(\sin\alpha + \cos\alpha) - y(\cos\alpha - \sin\alpha) = p$

Comparing these lines with $ax + by + p = 0$

We get

$$\frac{a}{\cos\alpha - \sin\alpha} = \frac{b}{\sin\alpha + \cos\alpha} = 1$$

$$\Rightarrow \quad a^2 + b^2 = 2$$

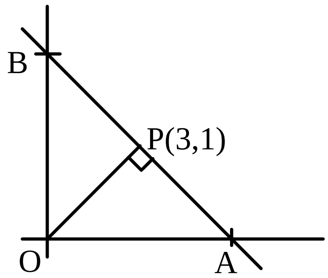
Or $$\frac{a}{\sin\alpha + \cos\alpha} = \frac{b}{\sin\alpha - \cos\alpha} = 1$$

$$\Rightarrow \quad a^2 + b^2 = 2$$

5. Line AB will be farther from origin if OP is right angle to the drawn line.

$$m_{OP} = \frac{1}{3}$$

$$\Rightarrow \quad m_{AB} = -3$$



Thus equation of AB is

$$(y-1) = -3(x-3)$$

$$\Rightarrow \quad A \equiv \left(\frac{10}{3} - 0\right) B \equiv (0,10)$$

$$\Rightarrow \quad \Delta_{OAB} = \frac{1}{2}(OA)(OB)$$

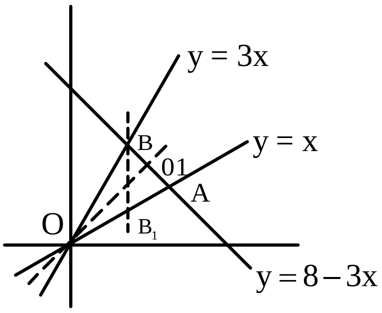
$$= \frac{1}{2}, \frac{10}{3}10 = \frac{100}{6}\text{sq.units}$$

6. $A \equiv (2,2), B \equiv \left(\frac{4}{3}, 4\right)$

Equation of bisector OO_1 is

$$\frac{y-3x}{\sqrt{10}} = -\frac{(y-x)}{\sqrt{2}}$$

$\Rightarrow \quad y\left(\frac{1}{\sqrt{5}}+1\right)-x\left(1+\frac{1}{\sqrt{5}}\right)=0$



Equation of BB_1 is

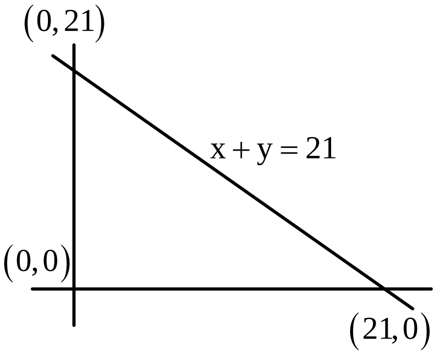
$$\frac{y+3x-8}{\sqrt{10}}=\frac{y-3x}{\sqrt{10}}$$

$\Rightarrow \quad x=\frac{4}{3}$

$\Rightarrow \quad \text{Incenter} \equiv \left(\frac{4}{3},\left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right)\frac{4}{3}\right)$

i.e., $\left(\frac{4}{3},\frac{2+2\sqrt{5}}{3}\right)$

7.



For any integral point (x, y) inside the triangle

$2 \le x+y \le 20$

$(1,1)(1,2)......(1,19) \rightarrow 19$ points

$(2,1)(2,2)......(2,18) \rightarrow 18$ points

$(3,1)(3,2)......(3,17) \rightarrow 17$ points

;

;

;

$(19,1) \rightarrow 1$ points

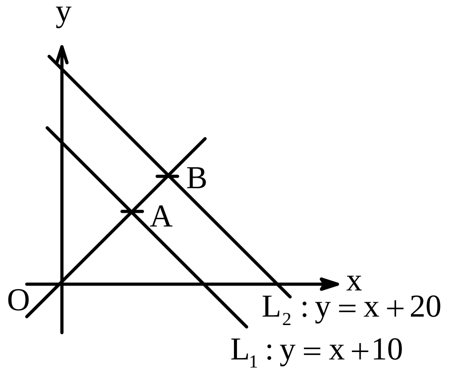
$\therefore$ Total number of Integral points in the interior of triangle are.

$1+2.....+19=\frac{19\times 20}{2}=190$

8. Let the parametric equation of drawn line is

$$\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r$$

$\Rightarrow$ $x = r\cos\theta, y = r\sin\theta$



Putting it in L_1 we get

$$r\sin\theta = r\cos\theta + 10$$

$\Rightarrow$ $\frac{1}{OA} = \frac{\sin\theta - \cos\theta}{10}$

Similarly putting the general point of drawn line in the equation of L_2, we get

$$\frac{1}{OB} = \frac{\sin\theta - \cos\theta}{20}$$

Let $P = (h, k)$ and $OP = r$

$\Rightarrow$ $r\cos\theta = h, r\sin\theta = k$,

We have $\frac{2}{r} = \frac{\sin\theta - \cos\theta}{10} + \frac{\sin\theta - \cos\theta}{20}$

$\Rightarrow$ $40 = 3r\sin\theta - 3r\cos\theta\theta$

$\Rightarrow$ $3y - 3x = 40$

9. $r^2 = \frac{10.20}{(\sin\theta - \cos\theta)^2}$

$\Rightarrow$ $(r\sin\theta - r\cos\theta)^2 = 200$

Thus locus is $(y - x)^2 = 200$

10. $\frac{1}{r^2} = \frac{(\sin\theta - \cos\theta)^2}{100} + \frac{(\sin\theta - \cos\theta)^2}{400}$

$\Rightarrow$ $400 = 5(4\sin\theta - r\cos\theta)^2$

Thus locus is

$$400 = 5(x - y)^2$$

i.e., $(x - y)^2 = 80$

11. $A_1 \equiv \left(-\frac{c_1}{2}, 0\right), A_2 \equiv \left(-\frac{c_2}{2}, 0\right), B_1 \equiv (0, c_1),\ B_2 \equiv (0, c_2)$

Equation of A_1B_2 is

$$-\frac{2x}{c_1} + \frac{y}{c_2} = 1$$

And equation of A_2B_1 is

$$-\frac{2x}{c_2} + \frac{y}{c_1} = 1$$

For the point of intersection of these lines, we have

$$y\left(\frac{1}{c_2} - \frac{1}{c_1}\right) + 2x\left(\frac{1}{c_2} - \frac{1}{c_1}\right) = 0$$

$\Rightarrow \quad y + 2x = 0$

12. Let $A \equiv (h, k)$

$$m_{AH} = \frac{k-4}{h-1},\ m_{OB} = \frac{4}{3}$$

$\Rightarrow \quad \frac{(k-4)}{(h-1)}\frac{4}{3} = -1$

$\Rightarrow \quad 4k + 3h = 19 \quad$ .... (i)

$$m_{OA} = \frac{k}{h}, m_{BH} = \frac{4-4}{3-1} = 0$$

Since OA and BH are mutually perpendicular, it implies that $h = 0$

Putting $h = 0$ in (i), we get

$$k = \frac{19}{4}$$

Thus coordinate of A is $\left(0, \frac{19}{4}\right)$

13. Let A_1 is the reflection of A in $y = x$

$\Rightarrow \quad A_1 = (4, 3)$

Now, $\quad PA + PB = A_1P + PB$

Which is minimum if A_1, P and B are collinear.

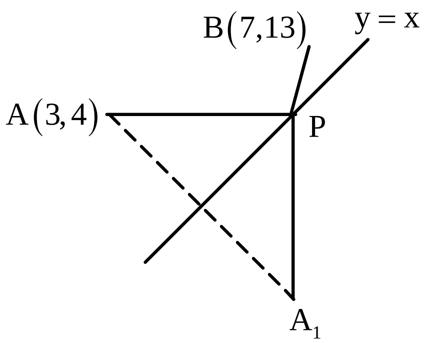
Equation of A_1B is

$$(y-3) = \frac{13-3}{7-4}(x-4)$$

$$3y = 10x - 31$$

Solving it with $y = x$, we get

$$P \equiv \left(\frac{31}{7}, \frac{31}{7}\right),$$



14. We have $|PA - PB| \le AB$

Thus for $|PA - PB|$ to be maximum, points A,

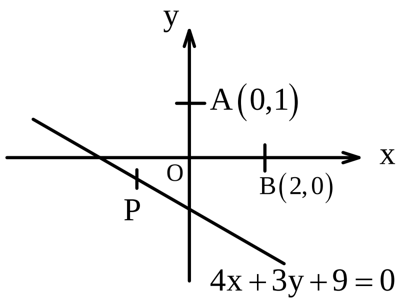
B and P must be collinear

Equation of AB is

$$x + 2y = 2$$

Solving it with given line, we get

$$P \equiv \left(-\frac{84}{5}, \frac{13}{5}\right)$$



15. Minimum value of $|PA - PB|$ is zero.

It can be altained if $PA = PB$

That means P must lie on the right bisector of AB

Equation of rightbisector of AB is

$$y - \frac{1}{2} = 2(x - 1)$$

i.e., $y = 2x - \frac{3}{2}$

Solving with given line, we get

$$P \equiv \left(-\frac{9}{20}, -\frac{12}{5}\right)$$

16. 'B' and 'C' will be the reflection of A in

$y + x = 0$ and $y - x = 0$ respectively.

Thus $B \equiv (-7, -5), C \equiv (7, 5)$

Hence equation of BC is

$$y - 5 = \frac{-5 - 5}{-7 - 7}(x - 7)$$

i.e., $14y = 10x$

17. Let the equation of chord be

$$y = mx + c$$

Combined equation of line joining the point of intersection with origin is

$$3x^2 - y^2 - 2(x - 2y)\left(\frac{y - mx}{c}\right) = 0$$

i.e., $x^2(3c + 2m) - y^2(c - 4) - 2xy(1 + 2m) = 0$

These lines will be mutually perpendicular if,

$$3c + 2m - c + 4 = 0$$

$\Rightarrow \quad 2m + 2c = -4$

$\Rightarrow \quad m + c = -2$

That means the chord $y = mx + c$ will always pass through the point $(1, -2)$

18. $y + 2 = 0$ and $x - a - b = 0$ intersect at $(a + b, 2)$

Thus lines will be concurrent if

$$2 = b(a + b)$$

19. $m_{AA_1} = \frac{O - 1}{\frac{b}{2} - 0} = \frac{2a}{b}$,

$$m_{BB_1} = \frac{\frac{a}{2} - 0}{0 - b} = -\frac{a}{2b}$$

Medians AA_1 and BB_1 will be mutually perpendicular if

$$-\frac{2a}{b} . -\frac{a}{2b} = 1$$

$\Rightarrow \quad a^2 + b^2 = 0$, which is not possible.

20. If the remaining vertex is (h, k) then

$h + 6 = 3 + 2$

$h = -1$

$k + 4 = 5 - 1$

$k = 0$

$(-1, 0)$

$h - 3 = -2 + 6 \equiv h = 7$

$h - 5 = 1 + 4 \equiv k = 10$

$h + 6 = -2 - 3 \equiv h = -11$

$k + 4 = 1 - 5 = k = -8$

21. Distance of all the points from (0, 0) are 5 units. That means circumcenter of the triangle formed by the given point is $(0, 0)$

If $G \equiv (h, k)$ be the centroid of triangle, then

$$3h = 3 + 5(\cos\theta + \sin\theta)$$

$$3k = 4 + 5(\sin\theta - \cos\theta)$$

$$3k = 4 + 5(\sin\theta - \cos\theta)$$

If $H(\alpha, \beta)$ be the orthocenter, then

$$OG : GH = 1 : 2$$

$\Rightarrow \quad \alpha = 3h, \beta = 3k$

$$\cos\theta + \sin\theta = \frac{\alpha - 3}{5}, \; \sin\theta - \cos\theta = \frac{\beta - 4}{5}$$

$\Rightarrow \quad \sin\theta = \dfrac{\alpha + \beta - 7}{10}, \cos\theta = \dfrac{\alpha - \beta + 1}{10}$

Thus, required locus of (α, β) is

$$(x + y - 7)^2 + (x - y + 1)^2 = 100$$

22. Let the lines represented by

$ax^2 + 2hxy + by^2 = 0$ be

$y = m_1 x$ and $y = m_2 x$

Then, $m_1 + m_2 == -\dfrac{2h}{b}, m_1 m_2 = \dfrac{a}{b}$

If theses lines are reflected about the x – axis, their equation becomes

$$y = m_1 x = 0, y + m_2 x = 0$$

And their combined equation is

$$(y + m_1 x)(y + m_2 x) = 0$$

$\Rightarrow \quad y^2 + xy(m_1 + m_2) + m_1 m_2 x^2 = 0$

i.e., $\quad by^2 - 2hxy + ax^2 = 0$

23. Combined equation of angle bisectors is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

If one bisector is $y = x$, then

$$x^2 - y^2 = 0$$

$\Rightarrow \quad a - b = 0$

24. $AB = 4, BC = 3$

$\Rightarrow \quad AC = 5$

$A \equiv (5\cos\theta, 0), C \equiv (0, 5\sin\theta)$

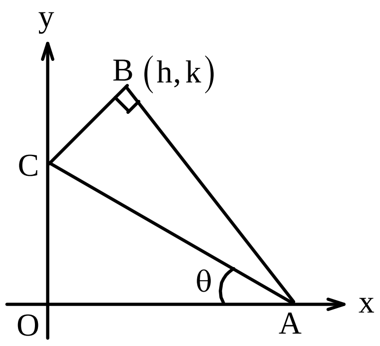
If BC and AB makes the angle θ_1 and θ_2 with x – axis, then

$\theta_1 = C - \theta,$

$\theta_2 = \pi - (A + \theta)$

$= \pi - \left(\frac{\pi}{2} - C + \theta\right)$

$= \frac{\pi}{2} + (C - \theta)$



Using parametric equation of lines for BC, we get

$$\frac{h}{\cos(c-\theta)} = \frac{k - 5\sin\theta}{\sin(c-\theta)} = 3 \quad \text{.......... (i)}$$

Similarly, using parametric equation of line for AB, we get

$$\frac{h - 5\cos\theta}{-\sin(c-\theta)} = \frac{k}{\sin(c-\theta)} = 4 \quad \text{......... (ii)}$$

From (i) and (ii) we get

$$\sin(c-\theta) = \frac{h}{3} = \frac{k}{4}$$

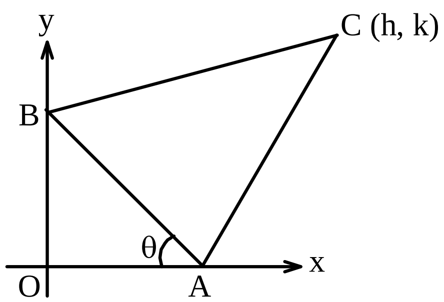
Thus, locus of 'B' is,

$4x = 3y$

25. $AB = BC = CA = 2$

Let $\angle BAO = \theta$

$\Rightarrow \quad A \equiv (2\cos\theta, 0), B \equiv (0, 2\sin\theta)$



BC makes an angle $\left(\frac{\pi}{3} - \theta\right)$ with x – axis and

AC makes an angle $\left(\pi-\left(\frac{\pi}{3}+\theta\right)\right)$ with x – axis.

If $C \equiv (h, k)$, then

$$\frac{h}{\cos\left(\frac{\pi}{3}-\theta\right)}=\frac{k-2\sin\theta}{\sin\left(\frac{\pi}{3}-\theta\right)}=2$$

And $$\frac{h-2\cos\theta}{-\cos\left(\frac{\pi}{3}+\theta\right)}=\frac{k}{\sin\left(\frac{\pi}{3}+\theta\right)}=2$$

$\Rightarrow$ $$2\cos\left(\frac{\pi}{3}-\theta\right)=h, 2\sin\left(\frac{\pi}{3}+\theta\right)=k$$

$\Rightarrow$ $$\cos\theta+\sqrt{3}\sin\theta=h$$

$$\sqrt{3}\cos\theta+\sin\theta=k$$

$\Rightarrow$ $$2\cos\theta=\left(\sqrt{3}k-h\right), 2\sin\theta=\left(h\theta-k\right)$$

Thus locus of (h, k) is

$$1=x^2+y^2-xy\sqrt{3}$$