CIRCLE SOLUTION

1. Since $y=|x|+c$ and $x^2+y^2-8|x|-9=0$ both are symmetrical about y-axis for $x>0, y=x+c.$

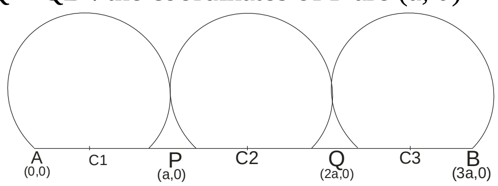
Equation of tangent to circle $x^2+y^2-8x-9=0$

Parallel to $y=x+c$ is $y=(x-4)+5\sqrt{1+1}$

$\Rightarrow \qquad y=x+\left(5\sqrt{2}-4, \infty\right)$

for no solution $c>5\sqrt{2}-4$; $\therefore c\,\varepsilon\left(5\sqrt{2}-4,\infty\right)$

2. Since AP = PQ = QB . the coordinates of P are (a, 0)



and of Q are (2a, 0) the centre of the circles on AP, PQ and QB as diameters are respectively $C_1\left(\frac{a}{2},0\right), C_2\left(\frac{3a}{2},0\right)$ and $C_3\left(\frac{5a}{2},0\right)$ and the radius of each one of them is $\left(\frac{a}{2}\right)$.

Hence, the equations of the circles with centre C_1, C_2 and C_3 are respectively.

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4};\left(x-\frac{3a}{2}\right)^2+y^2=\frac{a^2}{4}$$

And $\left(x-\frac{5a}{2}\right)^2+y^2=\frac{a^2}{4}.$

So that, if S(h, k) be any point on the locus, then

$$\left(h-\frac{a}{2}\right)^2+\left(h-\frac{3a}{2}\right)^2+\left(h-\frac{5a}{2}\right)^2+3\left(k^2-\frac{a^2}{4}\right)=b^2$$

$$3\left(h^2+k^2\right)-9ah+8a^2=b^2$$

3. Let $\phi(x,y)=x^2+y^2+2gx+2fy+c=0$

$\therefore \phi(0,\lambda)=0, \lambda^2+2f\lambda+c=0$

Have equal roots,

Then $2+2=-\frac{2f}{1}\, and\, 2\cdot 2=\frac{c}{1}$

$\therefore \qquad f=-2\, and\, c=4$

$and\, \phi(\lambda,0)=\lambda^2+2g\lambda+c=0$

$\therefore \qquad \lambda^2+2g\lambda+c=0$

$Here\, c=4 \qquad \therefore \lambda^2+2g\lambda+4=0$

Have roots 4/5,5

$\therefore \qquad \frac{4}{5}+5=2g$

$\Rightarrow \qquad g=-\frac{29}{10}$

$\therefore \qquad Centre = (-g, -f) = \left(\frac{29}{10}, 2\right)$

4. Circle is $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$$x^2 + y^2 = r^2$$

Equation of tangent at θ is

$$x\cos\theta + y\sin\theta = r \qquad \ldots(i)$$

and at $(\theta + \pi/3)$ *is*

$$x\cos(\theta+\pi/3) + y\sin(\theta+\pi/3) = r$$

$$\Rightarrow \; x\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) + y\left(\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) = r$$

$$\Rightarrow x\cos\theta + y\sin\theta - x\sqrt{3}\sin\theta + y\sqrt{3}\cos\theta = 2r$$

$$\Rightarrow \qquad r - \sqrt{3}(x\sin\theta - y\cos\theta) = 2r$$

or $\qquad x\sin\theta - y\cos\theta = -\frac{r}{\sqrt{3}} \qquad \ldots(ii)$

Squaring and adding Eqs. (i) and (ii), then we get

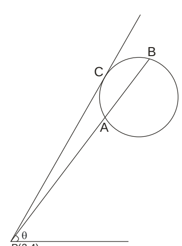
$$x^2 + y^2 = \frac{4r^2}{3}$$

$$\Rightarrow \qquad 3(x^2 + y^2) = 4r^2$$

5. Given, $(1+\alpha x)^n = 1 + 8x + 24x^2 + \ldots$

$$\Rightarrow \qquad 1 + n(\alpha x) + \frac{n(n-1)}{1\cdot 2}(\alpha x)^2 + \ldots$$

$$= 1 + 8x + 24x^2 + \ldots$$



Equating the coefficients of x and x^2, we get

$$n\alpha = 8, \frac{n\alpha(n\alpha - \alpha)}{1\cdot 2} = 24$$

or $\qquad \frac{8(8-\alpha)}{2} = 24$

$$\Rightarrow \qquad 8 - \alpha = 6$$

$$\therefore \qquad \alpha = 2 \; and \; n = 4$$

Equation of line is $\frac{x-2}{\cos\theta} = \frac{y-4}{\sin\theta} = r,$ then point

$(2 + r\cos\theta, 4 + r\sin\theta)$ lies on the circle $x^2 + y^2 = 4$

then $\left(2+r\cos\right)^2+\left(4+r\sin\right)^2=4$

or $r^2+4r\left(\cos\theta+2\sin\theta\right)+16=0$

$\therefore$ $PA\cdot PB=r_1r_2=\dfrac{16}{1}=16$

Alternative Method:

$$PA\cdot PB=\left(PC\right)^2$$
$$=2^2+4^2-4$$
$$=16$$

6. Let circle $x^2+y^2+2gx+2fy+c=0$ according to question

$2g\times 2+2f\times -3=c+9$

$\Rightarrow$ $4g-6f=c+9$ ...(i)

and $2g\times -\dfrac{5}{2}+2f\times 2=c+2$

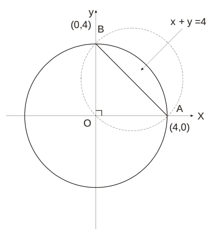
$-5g+4f=c+2$ ...(ii)

Subtracting Eqs. (ii) from (i),

$9g-10f=7$

or $9x-10y+7=0$

7. Equation of common chord is



$\left(x^2+y^2-16\right)-\left(x^2+y^2-4x-4y\right)=0$

$\Rightarrow$ $4x+4y-16=0$

$\therefore$ $x+y=4$

$\therefore$ $\angle AOB=\pi/2$

8. Centres of circles are

$C_1\left(-\lambda_1,0\right);C_2\left(-\lambda_1,0\right) and C_3\left(-\lambda_1,0\right)$

If $O\left(0,0,\right)$

then $OC_1=\lambda_1, OC_2=\lambda_2, OC_3=\lambda_3$

$\because OC_1, OC_2$ *and* OC_3 are in GP

$\because$ $\lambda_2^2=\lambda_1\lambda_3$ …(i)

Let any point on $x^2+y^2=c^2$ *is* $P\left(c\cos\alpha, c\sin\alpha\right)$

$\therefore$ Length of tangents are $PT_1=\sqrt{2\lambda_1\left(c\cos\alpha\right)}$

$PT_2=\sqrt{2\lambda_2\left(c\cos\alpha\right)}$

$PT_3=\sqrt{2\lambda_3\left(c\cos\alpha\right)}$

$\left(PT_2\right)^2=2\lambda_2\left(c\cos\alpha\right)$

$$= 2\sqrt{\lambda_1}\sqrt{\lambda_3}\sqrt{c\cos\alpha}\sqrt{c\cos\alpha} \text{ [from Eq. (i)]}$$

$$\sqrt{2\lambda_1(c\cos\alpha)}\sqrt{2\lambda_3(c\cos\alpha)}$$

$$= PT_1.PT_3$$

Hence PT_1, PT_2 *and* PT_3 are in GP.

9. Let the circle be $(x-h)^2+(y-k)^2=r^2$

then $r = \dfrac{|lh+mk+1|}{\sqrt{(l^2+m^2)}}$

or $l^2(h^2-r^2)+m^2(k^2-r^2)+2lmhk+2lh+2mk+1=0$

also $4l^2-5m^2+6l+1=0$

Comparing the coefficients of similar terms

$$h^2-r^2=4, k^2-r^2=-5, hk=0$$

$$2h=6, 2k=0$$

$\therefore k=0, h=3$ *and* $9-r^2=4$

$\therefore r=\sqrt{5}$

10. $\because f(x+y)=f(x)\cdot f(y)$ …(i)

$\because \quad f(1)=2$

In Eq. (i), Put $x=1, y=2,$

then $f(2)=f(1)=2^2$

Now in Eq. (i), $x=1, y=2,$ *then*

$$f(3)=f(1)\cdot f(2)=2\cdot 2^2=2^3$$

Hence $f(n)=n^2$

$\therefore \quad \alpha_n = f(n)=2^2 \forall n \in N$

$$(\alpha_1,\alpha_2)\equiv(2,4)$$

and $(\alpha_3,\alpha_4)\equiv(8,16)$

Equation of circle in diametric form is

$(x-2)(x-8)+(y-4)(y-16)=0$

11. $x^2+y^2-2px-2qy+q^2=0$

or $(x-p)^2+(y-q)^2$ …(i)

Since tangents are perpendicular , then locus of point of intersection of tangents is director circle.
Director circle of (i) is

$(x-p)^2+(y-q)^2=p^2+p^2$ …(ii)

Since point of intersection of tangent is (0,0), then from Eq.
(ii)

$$p^2 + q^2 = 2p^2$$

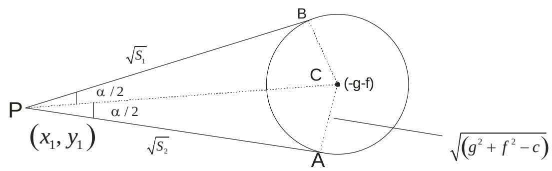
$\Rightarrow$ $$p^2 = q^2$$

or $$p = \pm q$$

12. $\therefore \cot \alpha / 2 = \dfrac{\sqrt{S_1}}{\sqrt{\left(g^2 + f^2 - c\right)}}$

$\therefore \tan \alpha / 2 = \dfrac{\sqrt{\left(g^2 + f^2 - c\right)}}{\sqrt{S_1}}$

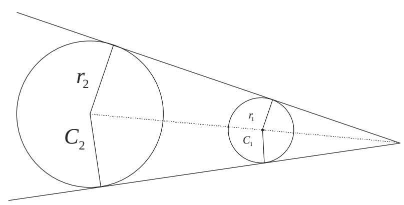
$\therefore \quad \alpha = 2\tan^{-1}\left(\dfrac{\sqrt{\left(g^2 + f^2 - c\right)}}{\sqrt{S_1}}\right)$



13. Centres and radii of the given circles are

$C_1(-7,2): r_1 = 5 \, and \, C_2(7,-2), r_2 = 9$

$\because \quad C_1C_2 = \sqrt{(14)^2 + (4)^2} = \sqrt{212} > r_1 + r_2$



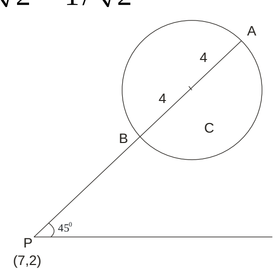
For common tangent tangent length of perpendicular from centre on tangent = radius

Of centre $C_1(-7,2) \, and \, r_1 = 5,$ then (b) and (c) are correct.

14. Slop of $PC = \dfrac{-3-2}{2-7} = 1$

If $\quad \tan\theta = 1 \because \theta = 45^0$

Equation of $PA \; is \dfrac{x-7}{1/\sqrt{2}} = \dfrac{y-2}{1/\sqrt{2}} = r$



$\therefore \left(7 + \dfrac{r}{\sqrt{2}}, 2 + \dfrac{r}{\sqrt{2}}\right)$ lie on circle,

$$then, \left(7+\frac{r}{\sqrt{2}}\right)^2 + \left(2+\frac{r}{\sqrt{2}}\right)^2 - 4\left(7+\frac{r}{\sqrt{2}}\right)^2$$

$$+6\left(7+\frac{r}{\sqrt{2}}\right)^2 - 3 = 0$$

$$\Rightarrow r^2 + 10\sqrt{2}r + 34 = 0$$

$$\therefore \quad r = -5\sqrt{2} \pm 4$$

$$\therefore Points \left(7+\frac{-5\sqrt{2}\pm 4}{\sqrt{2}}, 2+\frac{-5\sqrt{2}\pm 4}{\sqrt{2}}\right)$$

$$\Rightarrow \quad \left(2\pm 2\sqrt{2}, -3\pm 2\sqrt{2}\right)$$

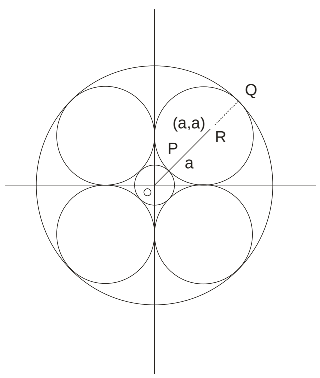
15. Radius of outer circle $= OR - a$

$$= \sqrt{a^2 + a^2} - a$$

$$= a\left(\sqrt{2}-1\right)$$

Radius of outer circle = OR + RQ

$$= a\sqrt{2} + a = a\left(\sqrt{2}+1\right)$$



16. $\because C_2$ is the director circle of C_1

$\because$ Equation of C_2 is

$$x^2 + y^2 = 2(2)^2 = 8$$

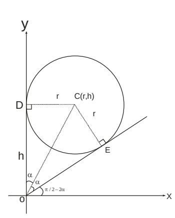
Again C_3 is the director circle of C_2, Hence the equation of C_3 is

$$x^2 + y^2 = 2(8) = 16$$

17. The given equation is

$$(x-r)^2 + (y-h)^2 = r^2$$

Tangents are x = 0



and $\quad y = x \tan\left(\frac{\pi}{2} - 2\alpha\right)$

$$= x\cot 2\alpha$$

$$= \frac{x\left(1-\tan^2\alpha\right)}{2\tan\alpha}$$

$$y = \frac{x\left(1-\frac{r^2}{h^2}\right)}{2\left(\frac{r}{h}\right)} \qquad \left(\because in\triangle ODC, \tan\alpha = \frac{r}{h}\right)$$

or $\left(h^2 - r^2\right)x - 2rhy = 0$

18. $\because 16m^2 = 8l + 1$

$\Rightarrow \quad 16\left(l^2 + m^2\right) = 16l^2 + 8l + 1$

$= \left(4l+1\right)^2$

or $4\sqrt{\left(l^2+m^2\right)} = |4l+1|$

or $\frac{|4l+1|}{\left(l^2+m^2\right)} = 4$

$\therefore$ Centre = (4,0) and radius = 4

Equation of circle is $\left(x-4\right)^2 + \left(y-0\right)^2 = 4^2$

$\Rightarrow \quad x^2 + y^2 - 8x = 0$

19. $\because 9l^2 + 16m^2 + 1 + 2 + 24lm + 6l + 8m$

$= 25\left(l^2 + m^2\right)$

$\Rightarrow \left(3l + 4m + 1\right)^2 = 25\left(l^2 + m^2\right)$

$\Rightarrow \frac{|3l+4m+1|}{l^2+m^2} = 5$

$\therefore$ Centre = (3,4) and radius = 5

$\Rightarrow$ Equation of circle is $\left(x-3\right)^2 + \left(y-4\right)^2 = 5^2 = 25$

$\therefore$ Equation of director circle is $\left(x-3\right)^2 + \left(y-4\right)^2 = 50$

or $x^2 + y^2 - 6x - 8y = 25$